Digraphs, details, and acyclic orientations

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Directed graph

Definition

A directed graph (or digraph) D has vertices $V(D)$ and directed arcs $A(D)$ connecting the vertices. It is simple if

- finite vertices
- no arc joining vertex to itself
- not multiple arcs going in same direction.

Note: Every arc must have a direction.

Example

$D =$ is a simple digraph.

Underlying graph

Definition

Let D be a digraph. Then the underlying graph of D, $U(D)$, is the graph obtained by removing all the arc directions.

Note: $U(D)$ simple $\Rightarrow D$ simple, but D simple $\not\Rightarrow U(D)$ simple.

Example

$D =$ is a simple digraph with $U(D) =$

Isomorphic

Definition

Let D_1 and D_2 be two digraphs.They are isomorphic if and only if $U(D_1) \cong U(D_2)$ and the isomorphism preserves the arc orderings.

Example

Degrees

Definition

Let D be a digraph.The number of arc ends meeting at a vertex v is called the degree, $deg(v)$. Arcs coming in is the indegree, $indeg(v)$. Arcs going out is the outdegree, $outdeg(v)$.

Note:

- $deg(v) = indeg(v) + outdeg(v)$.
- $indeg(v) = 0$ is a source, $outdeg(v) = 0$ is a sink.

Example

$D =$ has $indeg(v) =$, $outdeg(v) =$, $deg(v) =$

Adjacency matrix

Definition

Let D be a digraph with n vertices $v_1, v_2, \ldots, v_n$. If two vertices are joined by an arc they are adjacent. The adjacency matrix $A = (a_{ij})$ of D has

$$a_{ij} = \text{ no. arcs from } v_i \text{ to } v_j.$$

Note: Vertex v_i is a

- sink if row i is all 0,
- source if column i is all 0.

Example

$D =$ has $A = \begin{pmatrix} \quad \end{pmatrix}$

(Strongly) connected

Many concepts are similar to graphs, e.g. a directed path/cycle is a path/cycle whose directions you follow.

Definition

Let D be a digraph. It is connected if $U(D)$ is connected. It is strongly connected if it is connected and a directed path between every two vertices in both directions.

Example

$D =$

is connected but not strongly.

$\tilde{D} =$

is both connected and strongly.

Orientable

Definition

Let G be a graph. If we can turn edges into arcs so digraph is strongly connected, then G is orientable.

Note: If G is Eulerian then G is orientable - just mark the direction of travel on the Eulerian trail.

Example

$G =$

is orientable.

$H =$

is not orientable.

Orientations

Definition

Let G be a graph. If we turn edges into arcs, then the digraph D_G is an orientation of G.

If D_G has no directed cycles, then it is an acyclic orientation. The number of acyclic orientations is $a(G)$.

Example Orientations of K_3 are below, so $a(G) =$

Acyclic orientations

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Theorem (Stanley, 1973)

Let G be a simple graph with n vertices. Then

Proof idea About 30 mins, using a deletion/contraction for $a(G)$, and then an induction.

Example Let $G = K_3$.

In summary

- Digraphs and underlying graphs
- Details: isomorphic, adjacency matrix, directed paths and cycles
- Connected and strongly connected
- Orientable and Eulerian graphs
- Enumerating acyclic orientations

Thanks! See you next time!