Absolutely continuous spectrum for a random potential on a tree with strong transverse correlations and large weighted loops

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Abstract

We consider random Schrödinger operators on tree graphs and prove absolutely continuous spectrum at small disorder for two models. The first model is the usual binary tree with certain strongly correlated random potentials. These potentials are of interest since for complete correlation they exhibit localization at all disorders. In the second model we change the tree graph by adding all possible edges to the graph inside each sphere, with weights proportional to the number of points in the sphere.

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1. Introduction

Proving the existence of absolutely continuous spectrum for random Schrödinger operators at weak disorder remains a challenging problem. The extended states conjecture, asserting the existence of absolutely continuous spectrum at low disorder for the Anderson model on $\mathbb{Z}^d$, $d \geq 3$

remains the most important open problem in the field. When $\mathbb{Z}^d$ is replaced by the Bethe Lattice (or tree graph) this conjecture has been proved by Klein [K], extended and reproved by Aizenman, Sims and Warzel [ASW1], [ASW2], [ASW3], [AW1], and given yet another proof by the present authors [FHS2]. Our proof, which only applied to binary trees, has been simplified and extended by Halasan [H] to cover trees with higher branching number, and with additional vertices. (See also Spitzer [Sp].) Recent work on trees includes level statistics by Aizenman and Warzel [AW2] and localization respectively singular continuous spectrum by Breuer [B1] and [B2], and by Breuer and Frank [BF].

There is a large gap between the known results for the tree and the open problem on $\mathbb{Z}^d$. This present paper is an attempt to address some of the problems that would come up on $\mathbb{Z}^d$ in simpler models. The paper has two parts. In the first part we consider a binary tree with a transversely 2-periodic random potential. The potential is defined by choosing two values of the potential at random, independently for each sphere or level (that is, a set of vertices a fixed distance in the graph from the origin) in the tree. These two values are then repeated periodically across the sphere. The point of this model is that although the underlying graph is still a tree, we have negated some of the advantage of the exponential spreading of the tree.

In fact, such two-periodic potentials can exhibit either dense point spectrum or absolutely continuous spectrum. In our previous paper [FHS1], the values (q_1, q_2) were chosen close to $(\delta, -\delta)$ for $\delta > 0$. In this case we obtained a deterministic result proving existence of absolutely continuous spectrum. On the other hand, if (q_1, q_2) are chosen randomly on the diagonal $q_1 = q_2$ then the potential is radial, and this model is equivalent to a one-dimensional Anderson model that exhibits localization at all disorders.

We will prove that if the potentials (q_1, q_2) are sufficiently uncorrelated (see assumption (8) below) then there will be some absolutely continuous spectrum, as is the case for the Anderson model. However, since in some sense this model is so close to being one-dimensional, the proof has some features not appearing in [FHS2]. In both [FHS2] and the present paper, the proofs follow from an estimate of an average over potential values q of functions $\mu(z, q)$, similar in both models, that measure the contraction of a relevant map of the plane. We seek an estimate of the form $\int \mu(z, q)d\nu(q) < 1$ for z near the boundary at infinity. In [FHS2] we use the independence of the potentials across the sphere in proving that $\mu(z, 0)$ is already less than one. Then small values of q in the integral are handled by semi-continuity. In the present situation $\mu(z, q)$ for $q = 0$ is identically equal to one, and perturbations in q send it in both directions. Thus we must use cancellations in the integral over q in an essential way.

Our method extends to the case where the joint distributions are not identical, as long as they are all centered and satisfy certain uniform bounds. This is significant since in this case we lose the self-similarity that has been used in previous proofs.

Another obvious way that $\mathbb{Z}^d$ differs from the tree is in the presence of arbitrarily large loops. In the second part of this paper, we show how to introduce (weighted) loops with unbounded size into the model from the first part. We introduce connections between every pair of vertices in a given sphere, weighted to make the total weight of the added edges equal to one in each sphere. This is a sort of mean field interaction. These connections mean that when we remove the interior of some ball from the graph, the resulting exterior domain does not consist of disconnected pieces equivalent to the original graph, as is the case for the tree. Nevertheless, we can prove absolutely continuous spectrum for this model using results from the first part of this paper in a two-step procedure. To reduce the technical complication, we will only consider a Bernoulli distribution for the potentials in this section.

In the next section we review the basic set-up for calculating a diagonal matrix element of the Green's function for discrete random Schrödinger operators, using a decomposition of the graph and the corresponding sequence of forward Green's functions. In Section 3 we specialize to a tree model with a strongly transversely correlated random potential and present Theorem 2, the first main theorem. The bounds on the moment required in the proof of this theorem are given in Section 4 but the proof of the main technical Lemma 4 is postponed to Section 6. Section 5 deals with extensions and open problems related to our method of proof. The last two sections are devoted to the mean field tree model. Theorem 9 is our second main result. A proof of the main technical Lemma 12 needed for this theorem is relegated to Section 8.

2. Review of basic setup

Let (V, E) be a graph with vertex set V and edges $E \subseteq V \times V$, and let $\gamma : E \to \mathbb{R}^+$ be a bounded symmetric function. Let L be the Laplacian with matrix elements given by

$$L_{v,w} = \begin{cases} \gamma((v,w)) & \text{if } (v,w) \in E \\ 0 & \text{otherwise} \end{cases}.$$

We assume that the number of edges joining a vertex is uniformly bounded. Then L is a bounded, self-adjoint operator on $\ell^2(V)$.

Given a potential $q : V \to \mathbb{R}$, let Q be the operator of multiplication by q with matrix elements $Q_{v,w} = q(v)\delta_{v,w}$. We are interested in the spectrum of the discrete Schrödinger operator

$$H = L + Q$$

acting in $\ell^2(V)$. Let $0 \in V$ denote a distinguished vertex. We will study the spectral measure for H for the vector $\delta_0 \in \ell^2(V)$ given by

$$\delta_0(v) = \begin{cases} 1 & \text{if } v = 0 \\ 0 & \text{otherwise} \end{cases}$$

through its Borel transform given by the Green's function $G_0(\lambda) = \langle \delta_0, (H-\lambda)^{-1}\delta_0 \rangle$.

Our approach is based on a decomposition of V as a disjoint union and the corresponding direct sum decomposition of $\ell^2(V)$

$$V = \bigcup_{n=0}^{\infty} S_n, \quad \ell^2(V) = \bigoplus_{n=0}^{\infty} \ell^2(S_n).$$

We assume that $S_0 = \{0\}$ and that vertices in S_n are only connected to vertices in S_{n-1}, S_n and S_{n+1}. (We will take the sets S_n to be spheres containing all vertices a distance n in the graph from 0.) Then the block matrix forms of L and H have zeros away from the diagonal and first off-diagonal blocks.

$$L = \begin{bmatrix} D_0 & E_0^T & 0 & 0 & \cdots \\ E_0 & D_1 & E_1^T & 0 & \cdots \\ 0 & E_1 & D_2 & E_2^T & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad H = \begin{bmatrix} D_0+Q_0 & E_0^T & 0 & 0 & \cdots \\ E_0 & D_1+Q_1 & E_1^T & 0 & \cdots \\ 0 & E_1 & D_2+Q_2 & E_2^T & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

According to the formula for L, the matrix D_n is the Laplacian for the sphere S_n, while E_n has non-zero entries corresponding to the connections between S_n and S_{n+1}. Let P_n denote the projection of $\ell^2(V)$ onto $\ell^2(S_n)$ and define $P_{n,\infty} = \sum_{k=n}^{\infty} P_k$. Define $H_n = P_{n,\infty}\, H\, P_{n,\infty}$ and the forward Green's functions

$$G_n(\lambda) = P_n(H_n - \lambda)^{-1} P_n.$$

Each $G_n(\lambda)$ is a $d_n \times d_n$ matrix, where d_n is the number of vertices in S_n and lies in the Siegel upper half space $\mathbb{SH}_{d_n}$, that is, the space of symmetric $d_n \times d_n$ matrices with positive definite imaginary part; $\mathbb{H} := \mathbb{SH}_1$ is the usual complex upper half plane.

The forward Green's functions are related by the formula

$$G_n(\lambda) = \Phi_n(G_{n+1}, Q_n, \lambda), \tag{1}$$

where $\Phi_n : \mathbb{SH}_{d_{n+1}} \times S_{d_n} \times \mathbb{H} \to \mathbb{SH}_{d_n}$ is given by

$$\Phi_n(G_{n+1}, Q_n, \lambda) = -\left(E_n^T G_{n+1} E_n - D_n - Q_n + \lambda\right)^{-1}.$$

Here S_d is the set of $d \times d$ real symmetric matrices. To see this, note that $G_n(\lambda)$ is the top left corner block of

$$\begin{bmatrix} D_n + Q_n - \lambda & E_n^T \quad 0 \quad 0 \quad \cdots \\ E_n & \\ 0 & H_{n+1} - \lambda \\ \vdots & \end{bmatrix}^{-1}.$$

Thus, according to Schur's formula

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} (A - B^T C^{-1} B)^{-1} & (B^T C^{-1} B - A)^{-1} B^T C^{-1} \\ C^{-1} B (B^T C^{-1} B - A)^{-1} & (C - B A^{-1} B^T)^{-1} \end{bmatrix} \tag{2}$$

for the inverse of a symmetric block matrix we have

$$G_n(\lambda) = -\left([\, E_n^T \quad 0 \quad 0 \quad \cdots \,]\, (H_{n+1} - \lambda)^{-1} \begin{bmatrix} E_n \\ 0 \\ \vdots \end{bmatrix} - D_n - Q_n + \lambda \right)^{-1},$$

which implies (1).

Now suppose that the potential is chosen at random, independently for every sphere S_n according to a probability distribution N_n on $\mathbb{R}^{d_n}$. Then the matrices $G_n(\lambda)$ are random variables, distributed according to some measure $R_{n,\lambda}$ on $\mathbb{SH}_{d_n}$, and (1) implies that $R_{n,\lambda}$ is the push-forward of $R_{n+1,\lambda} \times N_n$ under Φ_n. This means that for every integrable function f on $\mathbb{SH}_{d_n}$

$$\int_{\mathbb{SH}_{d_n}} f(Z)\, dR_{n,\lambda}(Z) = \int_{\mathbb{SH}_{d_{n+1}}} \int_{\mathbb{R}^{d_n}} f(\Phi_n(Z, Q, \lambda))\, dN_n(Q)\, dR_{n+1,\lambda}(Z). \tag{3}$$

The measure in which we really are interested is $R_{0,\lambda}$, the distribution for G_0, which is a probability measure on $\mathbb{H}$. In our examples, we will use formula (3) to prove a bound of the form

$$\sup_{\substack{|\operatorname{Re}(\lambda)| \le \lambda_0 \\ 0 < \operatorname{Im}(\lambda) \le \epsilon}} \int_{\mathbb{H}} w^{1+\alpha}(z) dR_{0,\lambda}(z) < \infty, \tag{4}$$

where $\alpha > 0$ and $w(z)$ is a weight function satisfying

$$\operatorname{Im}(z) \le C w(z) \tag{5}$$

for z in a neighbourhood of the boundary at infinity $\partial_\infty \mathbb{H}$. In the upper half plane model of hyperbolic space $\mathbb{H}$, the boundary at infinity is $\mathbb{R} \cup \{i\infty\}$. A neighbourhood of $\partial_\infty \mathbb{H}$ is the complement of a closed bounded set in $\mathbb{H} \cup \partial_\infty \mathbb{H}$. Here and throughout the paper, C denotes a generic constant that may change from line to line. Notice that the integral in formula (4) is the expectation $\mathbb{E}\left[w^{1+\alpha}(G_0(\lambda))\right]$.

Lemma 1 *Suppose that (4) holds for some $\alpha > 0$ and some weight function $w(x)$ satisfying (5). Then the spectral measure μ_0 of which $G_0(\lambda)$ is the Borel transform is almost surely purely absolutely continuous in $(-\lambda_0, \lambda_0)$.*

Proof: (Following Klein [K] and Simon [Si].) By Fatou's lemma and (4)

$$\mathbb{E}\left(\liminf_{\epsilon \downarrow 0} \int_{-\lambda_0}^{\lambda_0} w^{1+\alpha}(G_0(x+i\epsilon)) dx \right) \le \liminf_{\epsilon \downarrow 0} \int_{-\lambda_0}^{\lambda_0} \mathbb{E}\left(w^{1+\alpha}(G_0(x+i\epsilon))\right) dx < C.$$

This implies that for almost every choice of potential

$$\liminf_{\epsilon \downarrow 0} \int_{-\lambda_0}^{\lambda_0} \left(\operatorname{Im}(G_0(x+i\epsilon))\right)^{1+\alpha} dx \le C \liminf_{\epsilon \downarrow 0} \int_{-\lambda_0}^{\lambda_0} \left(w(G_0(x+i\epsilon))\right)^{1+\alpha} dx < C.$$

So, for such a potential, there exists a sequence $\epsilon_n \downarrow 0$ such that

$$\sup_n \int_{-\lambda_0}^{\lambda_0} \left(\operatorname{Im}(G_0(x+i\epsilon_n))\right)^{1+\alpha} dx < C.$$

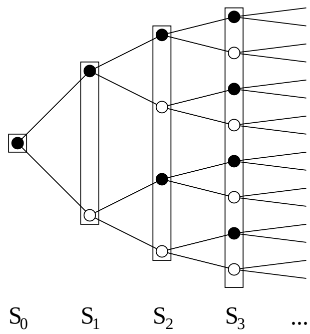
Then, since $\pi^{-1}\,\mathrm{Im}\,G_0(x+i\epsilon)dx$ converges to $d\mu_0(x)$ weakly (see [Si]) as $\epsilon \downarrow 0$ we find that for any compactly supported continuous function f

$$\begin{aligned}\left|\int_{-\lambda_0}^{\lambda_0} f(x)d\mu_0(x)\right| &= \lim_{n\to\infty} \pi^{-1}\left|\int_{-\lambda_0}^{\lambda_0} f(x)\,\mathrm{Im}\,G_0(x+i\epsilon_n)dx\right| \\ &\le \limsup_{n\to\infty} \pi^{-1}\left[\int_{-\lambda_0}^{\lambda_0} |f(x)|^q dx\right]^{1/q}\left[\int_{-\lambda_0}^{\lambda_0} (\mathrm{Im}\,G_0(x+i\epsilon_n))^{1+\alpha}\,dx\right]^{1/(1+\alpha)} \\ &\le C\|f\|_q.\end{aligned}$$

Here q is the dual exponent to $1+\alpha$ in Hölder's inequality. This implies that $d\mu_0(x) = g(x)dx$ for some $g \in L^{1+\alpha}$ and completes the proof. □

3. A binary tree with transversely 2-periodic potentials

We now specialize to a binary tree.



Rooted binary tree with transversely 2-periodic potential.

For a tree, the forward Green's functions are diagonal, and with

$$G_{n+1}(\lambda) = \mathrm{diag}[z_1, z_2, \ldots, z_{2^{n+1}}],$$
$$Q_n = \mathrm{diag}[q_1, q_2, \ldots, q_{2^n}],$$

we have

$$\Phi_n(G_{n+1}, Q_n, \lambda) = \mathrm{diag}\left[\frac{-1}{z_1+z_2+\lambda-q_1}, \ldots, \frac{-1}{z_{2^{n+1}-1}+z_{2^{n+1}}+\lambda-q_{2^n}}\right].$$

To define a two-periodic potential we choose for each sphere (except the root) two potential values $\mathbf{q} = (q_1, q_2)$ at random, independently for each sphere, according to an identical joint distribution ν. In the diagram, the spheres are outlined by boxes. For each sphere (except the first), after choosing $\mathbf{q} = (q_1, q_2)$, we set the potential at all the black vertices equal to q_1 and the potential at all the white vertices equal to q_2. The potential value at 0 is chosen according to some single site distribution $\nu_{(0)}$.

We make the following assumptions about this distribution ν. The distribution has bounded support:

$$\nu \text{ is supported in } \{\mathbf{q} = (q_1, q_2) : |q_1| \le 1, |q_2| \le 1\}. \tag{6}$$

The distribution is centred on zero:

$$\int (q_1 + q_2) d\nu(\mathbf{q}) = 0. \tag{7}$$

Let $c_{ij} = \int q_i q_j d\nu(\mathbf{q})$. Then

$$c = c_{11} + c_{22} > 0 \quad \text{and} \quad \delta = \frac{2c_{12}}{c_{11} + c_{22}} < 1/2. \tag{8}$$

The first inequality in (8) simply says that q is not identically zero. The second is a bound on the correlation. Completely correlated potentials (that is, the one-dimensional case where the spectrum is localized) would correspond to $\delta = 1$.

To adjust the disorder, we multiply the potential by a coupling constant $a > 0$ and study the Schrödinger operator $H_a = L + a\,Q$. This amounts to replacing ν with the scaled distribution ν_a satisfying

$$\int f(\mathbf{q})\, d\nu_a(\mathbf{q}) = \int f(a\mathbf{q})\, d\nu(\mathbf{q}).$$

The scaled distribution ν_a is supported in $\{\mathbf{q} = (q_1, q_2) : |q_1| \le a, |q_2| \le a\}$.

We can now formulate the main theorem for this section.

Theorem 2 *Let $\nu_{(0)}$ be a probability measure of bounded support for the potential at the root, let ν be a probability measure on $\mathbb{R}^2$ satisfying (6), (7) and (8) and let H_a be the random discrete Schrödinger operator on the binary tree corresponding to the transversely two-periodic potential defined by the scaled distribution ν_a. There exists $\lambda_0 \in (0, 2\sqrt{2})$ such that for sufficiently small a the spectral measure for H_a corresponding to δ_0 has purely absolutely continuous spectrum in $(-\lambda_0, \lambda_0)$.*

For a two-periodic potential, the formula (3) can be simplified. In this case the measure N_n is independent of n and concentrated on the two-dimensional hyperplane where $q_1 = q_3 = q_5 = \cdots$ and $q_2 = q_4 = q_6 = \cdots$. Thus, introducing a coupling constant a, the measure N_n is a product of ν_a with delta functions for the hyperplane. For these potentials the diagonal entries of $G_n(\lambda)$ exhibit the same symmetry as the potentials, so the probability distribution for $G_n(\lambda)$ is determined by the joint distribution $r_{a,\lambda}$ for (z_1, z_2), which also is independent of n. With this notation, the formula (3) can be written

$$\begin{aligned}
&\int_{\mathbb{H}\times\mathbb{H}} f(z_1, z_2)\, dr_{a,\lambda}(z_1, z_2) \\
&\qquad = \int_{\mathbb{H}\times\mathbb{H}\times\mathbb{R}^2} f\left(-\frac{1}{z_1 + z_2 + \lambda - q_1}, -\frac{1}{z_1 + z_2 + \lambda - q_2} \right) d\nu_a(\mathbf{q})\, dr_{a,\lambda}(z_1, z_2).
\end{aligned}$$

It is convenient to introduce a new random variable $u = z_1 + z_2 + \lambda$ for every sphere except the first. Let $\rho_{a,\lambda}$ denote the distribution on $\mathbb{H}$ for u. Then, taking $f(z_1, z_2) = g(z_1 + z_2 + \lambda)$ in the formula above we obtain our main recursion formula

$$\int_{\mathbb{H}} g(u)\, d\rho_{a,\lambda}(u) = \int_{\mathbb{H}\times\mathbb{R}^2} g(\phi_{\mathbf{q},\lambda}(u))\, d\nu_a(\mathbf{q})\, d\rho_{a,\lambda}(u), \tag{9}$$

where

$$\phi_{\mathbf{q},\lambda}(u) = -\frac{1}{u - q_1} - \frac{1}{u - q_2} + \lambda. \tag{10}$$

A source of difficulty is the singular behaviour of $\phi_{\mathbf{q},\lambda}$ near the diagonal of $\mathbf{q}$. When $q_1 = q_2$, (and $\mathrm{Im}(\lambda) \geq 0$) then $\phi_{\mathbf{q},\lambda}$ is a linear fractional transformation that defines an injective map from $\mathbb{H}$ to $\mathbb{H}$. In fact, if $\lambda \in \mathbb{R}$ the map is a hyperbolic isometry. However, as soon as $q_1 \neq q_2$ the map $\phi_{\mathbf{q},\lambda}$ covers $\mathbb{H}$ twice. This can be seen even when we only consider real values of u. In this case $\phi_{\mathbf{q},\lambda}(u)$ ranges over all of $\mathbb{R}$ for u in the interval (q_1, q_2) (supposing for the moment that $q_1 < q_2$). This interval shrinks and then disappears as q_1 approaches q_2.

We now introduce the weight function $\mathrm{cd}(u)$. For $\lambda \in (-2\sqrt{2}, 2\sqrt{2})$ the fixed point solution of $u \mapsto \phi_{\mathbf{0},\lambda}(u)$ is $u_\lambda = \lambda/2 + i\sqrt{2 - \lambda^2/4}$. Define

$$\mathrm{cd}(u) = \frac{|u - u_\lambda|^2}{\mathrm{Im}(u)}. \tag{11}$$

Our goal is to bound the moment

$$M_{a,\alpha,\lambda} = \int_{\mathbb{H}} \mathrm{cd}(u)^{1+\alpha} d\rho_{a,\lambda}(u). \tag{12}$$

Given Lemma 1, such a bound for $R_{0,\lambda}$ in place of $\rho_{a,\lambda}$ will provide a proof of Theorem 2. This is done in the following lemma.

Lemma 3 *Let $\nu_{(0)}$ be a probability measure of bounded support for the potential at the root, and suppose that*

$$\sup_{\substack{|\operatorname{Re}\lambda| \leq \lambda_0 \\ 0 < \operatorname{Im}\lambda \leq \epsilon}} M_{a,\alpha,\lambda} < C$$

for some positive a, α and ϵ. Then the spectral measure for δ_0 corresponding to the transversely two-periodic random potential with coupling constant a has purely absolutely continuous spectrum in $[-\lambda_0, \lambda_0]$.

Proof: Let $w(z) = |z - i|^2/\operatorname{Im}(z)$. The recursion formula (3) for the first level implies

$$\begin{aligned}\int_{\mathbb{H}} w(z)^{1+\alpha}\, dR_{0,\lambda}(z) &= \int_{\mathbb{H}\times\mathbb{R}} w(-(u-q)^{-1})^{1+\alpha}\, d\nu_{(0)}(q)\, d\rho_{a,\lambda}(u) \\ &\leq \left(\sup_{\mathbb{H}} \int_{\mathbb{R}} \left(\frac{w(-(u-q)^{-1})}{\mathrm{cd}(u)+1} \right)^{1+\alpha} d\nu_{(0)}(q) \right) (M_{a,\alpha,\lambda} + 1),\end{aligned}$$

so the lemma follows from Lemma 1 and the bound

$$\sup_{\mathbb{H}} \int_{\mathbb{R}} \left(\frac{w(-(u-q)^{-1})}{\mathrm{cd}(u)+1} \right)^{1+\alpha} d\nu_{(0)}(q) \leq C$$

by our assumption on $\nu_{(0)}$. □

4. Bounding $M_{a,\alpha,\lambda}$

Lemma 3 shows that our main theorem follows from a bound for $M_{a,\alpha,\lambda}$. We now explain how we can obtain such a bound. Beginning with (12) we introduce a cutoff function $\chi, 0 \leq \chi \leq 1$ with support in a neighbourhood of the boundary at infinity of $\mathbb{H}$, and with $\chi = 1$ near infinity. Since cd is bounded on the compact support of $1-\chi$,

$$M_{a,\alpha,\lambda} \leq \int_{\mathbb{H}} \chi(u)\ \mathrm{cd}(u)^{1+\alpha} d\rho_{a,\lambda}(u) + C,$$

where C only depends on the support of χ. Now we apply the recursion formula (9) to conclude

$$M_{a,\alpha,\lambda} \leq \int_{\mathbb{H}} \int_{\mathbb{R}^2} \chi(\phi_{\mathbf{q},\lambda}(u))\ \mathrm{cd}(\phi_{\mathbf{q},\lambda}(u))^{1+\alpha}\ d\nu_a(\mathbf{q}) d\rho_{a,\lambda}(u) + C.$$

Since the image of $\phi_{\mathbf{q},\lambda}(u)$ as $\mathbf{q}$ ranges over the support of ν_a, λ ranges over the rectangle $|\operatorname{Re}(\lambda)| \leq \lambda_0$, $0 \leq \operatorname{Im}(\lambda) \leq \epsilon$ and u ranges over the support of $1-\chi$ is compact, the function $\mathrm{cd}(\phi_{\mathbf{q},\lambda}(u))$ is bounded there, and we may again insert a cutoff and conclude

$$M_{a,\alpha,\lambda} \leq \int_{\mathbb{H}} \int_{\mathbb{R}^2} \chi(u)\ \chi(\phi_{\mathbf{q},\lambda}(u))\ \mathrm{cd}(\phi_{\mathbf{q},\lambda}(u))^{1+\alpha}\ d\nu_a(\mathbf{q}) d\rho_{a,\lambda}(u) + C. \tag{13}$$

The constant C is different from the previous equation, but can still be taken to be independent of λ in the range of values we are considering.

Here is the essential idea of our argument. Introduce

$$\mu_{\mathbf{q},\lambda}(u) = \frac{\mathrm{cd}(\phi_{\mathbf{q},\lambda}(u))}{\mathrm{cd}(u)} = \frac{|(2u-q_1-q_2)u_\lambda - 2(u-q_1)(u-q_2)|^2}{2(|u-q_1|^2+|u-q_2|^2)|u-u_\lambda|^2} \tag{14}$$

and the averaged version

$$\overline{\mu}_{a,\alpha,\lambda}(u) = \int_{\mathbb{R}^2} \mu_{\mathbf{q},\lambda}^{1+\alpha}(u)\ d\nu_a(\mathbf{q}).$$

Then (13) implies

$$M_{a,\alpha,\lambda} \leq \int_{\mathbb{H}} \chi(u)\, \overline{\mu}_{a,\alpha,\lambda}(u)\ \mathrm{cd}(u)^{1+\alpha} d\rho_{a,\lambda}(u) + C.$$

So if we knew that $\overline{\mu}_{a,\alpha,\lambda}(u) \leq 1-\epsilon_1$ on $\mathrm{supp}(\chi)$ for a suitable range of λ, then we would obtain $M_{a,\alpha,\lambda} \leq (1-\epsilon_1) M_{a,\alpha,\lambda} + C$ which gives the desired bound on $M_{a,\alpha,\lambda}$. Notice that the averaging over $\mathbf{q}$ is essential for obtaining such a bound, since $\mu_{\mathbf{0},\lambda}(u) = 1$.

Also note that $\mu_{\mathbf{q},\lambda}(u)$ is continuous as u and λ approach the real axis, except at $u = q_1 = q_2$. This includes $u = i\infty$, by which we mean continuity as $w \to 0$ when we set $u = -1/w$. At the singular point we can define $\mu_{\mathbf{q},\lambda}(u)$ to be supremum of all possible limits. In this way we can extend $\mu_{\mathbf{q},\lambda}(u)$ to an upper semi-continuous function whose domain includes real values of u and λ.

Here is the bound for $\overline{\mu}_{a,\alpha,\lambda}(u)$. This is the main technical result in the first part of the paper.

Lemma 4 *Suppose that ν is a probability measure on $\mathbb{R}^2$ satisfying (6), (7) and (8). Assume u and λ are real, $|\lambda| < 2\sqrt{2}$ and a and R are positive real numbers satisfying $R \geq 2$ and $aR \leq 1/4$. Then there exist positive constants C_i such that with c and δ defined by (8)*

$$\overline{\mu}_{a,\alpha,\lambda}(u) \leq \begin{cases} 1 - \dfrac{c(1-\delta)}{20R^2} + C_1 aR + C_2\alpha & \text{for } |u| \leq aR \\ 1 - \dfrac{a^2 c}{|u|^2}\left(\dfrac{p(u,\lambda,\delta)}{2|u-u_\lambda|^2} - \left(\dfrac{C_3}{R} + C_4\alpha\right)\right) & \text{for } |u| \geq aR \end{cases}, \tag{15}$$

where

$$p(u,\lambda,\delta) = (1-2\delta)u^2 - (1-\delta)\lambda u + 1 - \delta.$$

This lemma is proved in a separate section. When $|u| \to \infty$ the bound tends to 1, so this bound alone is not sufficient. To procede we must iterate the procedure leading to (13). Starting with (13) (with $\mathbf{q}$ replaced by $\mathbf{q}_1$) we apply (9) to arrive at

$$\begin{aligned} M_{a,\alpha,\lambda} &\leq \int_{\mathbb{H}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \chi(u)\, \chi(\phi_{\mathbf{q}_1,\lambda}(u))\, \mathrm{cd}(\phi_{\mathbf{q}_1,\lambda}\circ\phi_{\mathbf{q}_2,\lambda}(u))^{1+\alpha}\, d\nu_a(\mathbf{q}_1) d\nu_a(\mathbf{q}_2)\, d\rho_{a,\lambda}(u) + C \\ &= \int_{\mathbb{H}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \chi(u)\mu^{1+\alpha}_{\mathbf{q}_1,\lambda}(u)\, \chi(\phi_{\mathbf{q}_1,\lambda}(u))\mu^{1+\alpha}_{\mathbf{q}_2,\lambda}(\phi_{\mathbf{q}_1,\lambda}(u)) d\nu_a(\mathbf{q}_1) d\nu_a(\mathbf{q}_2) d\rho_{a,\lambda}(u) + C. \end{aligned} \tag{16}$$

In view of Lemma 3, the following lemma will complete the proof of Theorem 2.

Lemma 5 *Suppose that ν satisfies (6), (7) and (8). Then there exists $\lambda_0 \in (0, 2\sqrt{2})$ such that for small enough a and ϵ*

$$\sup_{\substack{|\operatorname{Re}\lambda| \leq \lambda_0 \\ 0 < \operatorname{Im}\lambda \leq \epsilon}} M_{a,\alpha,\lambda} < C.$$

Proof: Let

$$m_{a,\alpha,\lambda}(u) = \chi(u) \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mu^{1+\alpha}_{\mathbf{q}_1,\lambda}(u)\, \chi(\phi_{\mathbf{q}_1,\lambda}(u))\mu^{1+\alpha}_{\mathbf{q}_2,\lambda}(\phi_{\mathbf{q}_1,\lambda}(u)) d\nu_a(\mathbf{q}_1) d\nu_a(\mathbf{q}_2).$$

Given (16), it suffices to show that there exists $\lambda_0 \in (0, 2\sqrt{2})$ and $\epsilon_1 > 0$ so that

$$m_{a,\alpha,\lambda}(u) \leq 1 - \epsilon_1 \tag{17}$$

for all u in a neigbourhood of infinity in $\mathbb{H}$ and for a and α sufficiently small. An obvious estimate for $m_{a,\alpha,\lambda}(u)$ is

$$m_{a,\alpha,\lambda}(u) \leq \chi(u)\, \overline{\mu}_{a,\alpha,\lambda}(u) \sup_{\mathbf{q}_1 \in \operatorname{supp}(\nu_a)} \Big[\chi(\phi_{\mathbf{q}_1,\lambda}(u))\, \overline{\mu}_{a,\alpha,\lambda}(\phi_{\mathbf{q}_1,\lambda}(u))\Big]. \tag{18}$$

We begin by choosing λ_0 with

$$\lambda_0 < 2\sqrt{\frac{1-2\delta}{1-\delta}}.$$

Then a simple calculation shows that the polynomial $p(u, \lambda, \delta)$ in Lemma 4 is bounded below

$$p(u, \lambda, \delta) \geq p_0 > 0$$

for all $u \in \mathbb{R}$ and λ with $|\lambda| \leq \lambda_0$. Choosing R sufficiently large and α suffciently small we can simplify the estimate in Lemma 4 to read

$$\overline{\mu}_{a,\alpha,\lambda}(u) \leq \begin{cases} 1 - 2\epsilon_2 + C_1 aR & \text{for } |u| \leq aR \\ 1 - \dfrac{a^2\epsilon_3}{|u|^2} & \text{for } |u| \geq aR \end{cases}$$

for some $\epsilon_2, \epsilon_3 > 0$ and for all $u \in \partial_\infty \mathbb{H} = \mathbb{R} \cup \{i\infty\}$. Then, choosing a small (depending on R) we obtain

$$\overline{\mu}_{a,\alpha,\lambda}(u) \leq \begin{cases} 1 - \epsilon_2 & \text{for } |u| \leq aR \\ 1 - \dfrac{a^2\epsilon_3}{|u|^2} & \text{for } |u| \geq aR \end{cases}. \tag{19}$$

In particular, $\overline{\mu}_{a,\alpha,\lambda}(u) \leq 1$ for all $u \in \partial_\infty \mathbb{H}$. By upper semi-continuity of $\overline{\mu}$, we can extend this estimate to a neighbourhood of $\partial_\infty \mathbb{H}$ to conclude

$$\chi(u)\,\overline{\mu}_{a,\alpha,\lambda}(u) \leq 1 + \epsilon_4, \tag{20}$$

where ϵ_4 can be made arbitrarily small by shrinking the support of χ.

To estimate the right side of (18) we consider u in two regions. The first region are the points near $u \in \mathbb{R}$ with $|u| \leq C$. For these points, the estimate (19) and upper semi-continuity of $\overline{\mu}_{a,\alpha,\lambda}(u)$ imply

$$\overline{\mu}_{a,\alpha,\lambda}(u) \leq 1 - \epsilon_5$$

for some $\epsilon_5 > 0$. This, combined with (20), where we have shrunk the support of χ to make ϵ_4 sufficiently small, proves (17) for these values of u.

On the other hand, if u is in the region near $u \in \mathbb{R}$ with $|u| \geq C$ (including $i\infty$) then u is bounded away from the singularity of $\phi_{\mathbf{q}_1,\lambda}(u)$ for $\mathbf{q}_1 \in \mathrm{supp}(\nu_a)$, so for these values of u and small $\mathbf{q}_1$, the values of $\phi_{\mathbf{q}_1,\lambda}(u)$ are close to $\phi_{\mathbf{0},\lambda}(u)$ and therefore $|\phi_{\mathbf{q}_1,\lambda}(u)|$ is uniformly bounded. This means we can exchange the roles of the two factors in (18) and obtain (17) for these values of u as well. $\square$

5. Extensions and open problems

For $\delta = 0$, that is, when the random variables q_1 and q_2 are independent, our result gives $\lambda_0 = 2$. An obvious question is "How large can λ_0 actually be?". When λ_0 is larger than 2 the polynomial $p(u, \lambda, \delta)$ in (15) changes sign so the estimate for $\overline{\mu}_{a,\alpha,\lambda}(u)$ goes above 1 for some values of u. However, the product on the right side of (18) remains bounded below 1 if λ_0 is only slightly larger than 2, since the second term in the product compensates. So, our proof can accommodate λ_0 slightly larger than 2. To push λ_0 even higher, we can consider iterating the procedure leading to (16) an arbitrary number of times. This would presumably allow even larger values of λ_0 at the expense of more complicated proofs. The determination of the exact range of absolutely continuous spectrum (as indeed the question of band-edge localization for this model) remains open.

At first glance, it appears that the assumption that the distributions ν_a are identical for each sphere seems essential. Dropping it means that we lose self-similarity in the tree. However, in fact it is possible to handle the case where the distribution for the nth sphere $\nu_{a,n}$ can depend on n, provided that each distribution satisfies the assumptions (6), (7) and (8). Then the distributions $\rho_{a,\lambda,n}$ and the moments $M_{a,\alpha,\lambda,n}$ also vary from sphere to sphere. In this setup we are interested in $M_{a,\alpha,\lambda,1}$. The methods in this paper (with two iterations) can then be used to show that for suitable a, α and λ

$$M_{\lambda,n} \leq (1-\epsilon)M_{\lambda,n+2} + C. \tag{21}$$

(We have dropped the a and α subscripts.) Here ϵ and C are positive constants that are independent of n and λ. Iterating this bound N times gives

$$\begin{aligned} M_{\lambda,1} &\leq (1-\epsilon)^N M_{\lambda,1+2N} + C\left(\sum_{k=1}^{N-1}(1-\epsilon)^k\right) \\ &\leq (1-\epsilon)^N M_{\lambda,1+2N} + \frac{C}{\epsilon}. \end{aligned}$$

This estimate may appear useless, but for $\mathrm{Im}(\lambda) > 0$ we actually have an n independent (but λ dependent!) bound on $M_{\lambda,n}$, because the support of $\rho_{a,\lambda,n}$ is contained in a λ dependent compact set. Hence we obtain

$$M_{\lambda,1} \leq (1-\epsilon)^N C_\lambda + \frac{C}{\epsilon}$$

and we may send $N \to \infty$ to obtain the desired bound on $M_{\lambda,1}$.

6. Proof of Lemma 4

The goal of this section is to prove the estimates in Lemma 4 on μ defined by (14) for u and λ real. Notice that when $\lambda \in \mathbb{R}$ and $|\lambda| < 2\sqrt{2}$ then $\mathrm{Im}(u_\lambda) > 0$ and $|u_\lambda|^2 = 2$.

We will blow up the singularity on the diagonal by introducing polar co-ordinates r and ω_i, $i = 1, 2$ defined by

$$u - q_1 = r\omega_1, \quad u - q_2 = r\omega_2, \quad \omega_1^2 + \omega_2^2 = 1.$$

We begin with the estimate for $|u|$ small.

Lemma 6 *Suppose $|\lambda| < 2\sqrt{2}$, $|q_i| \le a$ and $|u| \le aR$ where $R \ge 2$ and $aR \le 1/4$. Then*

$$\mu_{\mathbf{q},\lambda}(u) \le \frac{|\omega_1 + \omega_2|^2}{2} + CaR.$$

Proof: We can write

$$\mu_{\mathbf{q},\lambda}(u) = \frac{|(\omega_1 + \omega_2)u_\lambda + 2r\omega_1\omega_2|^2}{2|u - u_\lambda|^2}.$$

We have

$$\frac{2}{|u - u_\lambda|^2} = \frac{2}{2 - \lambda u + u^2} \le \frac{1}{1 - \lambda u/2} \le 1 + |\lambda||u|$$

since $|\lambda u/2| \le 1/2$ and $(1 - x)^{-1} \le 1 + 2|x|$ for $|x| \le 1/2$. Next, we have

$$\frac{|(\omega_1 + \omega_2)u_\lambda + 2r\omega_1\omega_2|^2}{4} \le \frac{|\omega_1 + \omega_2|^2}{2} + r + r^2/4,$$

since $|\omega_1 + \omega_2| < \sqrt{2}$, $|\omega_1\omega_2| \le 1/2$. With our bounds on q_i and R we have

$$r^2 = |u - q_1|^2 + |u - q_2|^2 \le 2a^2(1 + R)^2.$$

Combining these estimates completes the proof. $\square$

Now we turn to the estimate for large $|u|$.

Lemma 7 *Suppose $|\lambda| < 2\sqrt{2}$, $|q_i| \le a$ and $|u| \ge aR$ where $R \ge 2$. Then*

$$\mu_{\mathbf{q},\lambda}(u) \le 1 + \frac{1}{u}\langle \mathbf{l}, \mathbf{q}\rangle - \frac{1}{u^2}\langle \mathbf{q}, (Q - C/R)\mathbf{q}\rangle$$

with

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},$$
$$\mathbf{l} = \frac{-2u^2 + \lambda u}{2|u - u_\lambda|^2}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$Q = \frac{1}{2|u - u_\lambda|^2}\begin{bmatrix} u^2 - \lambda u + 1 & -2u^2 + \lambda u - 1 \\ -2u^2 + \lambda u - 1 & u^2 - \lambda u + 1 \end{bmatrix}.$$

The constant $C = C_1/(1 - \lambda^2/8) + C_2$ where C_1 and C_2 are some (explicitly computable positive) numbers.

Proof: Let $\delta_i = q_i/u$ and note that $|\delta_i| < 1/R$. We can write

$$\mu_{\mathbf{q},\lambda}(u) = \frac{1}{4|u - u_\lambda|^2}|2(u - u_\lambda) - (\delta_1 + \delta_2)(2u - u_\lambda) + 2\delta_1\delta_2 u|^2 \frac{2}{|1 - \delta_1|^2 + |1 - \delta_2|^2}. \qquad (22)$$

The third term on the right can be written

$$\frac{2}{|1-\delta_1|^2+|1-\delta_2|^2} = \frac{1}{1-(\delta_1+\delta_2)+(\delta_1^2+\delta_2^2)/2}.$$

If $x \le \delta < 1$ then $(1-x)^{-1} \le 1+x+(1+\delta/(1-\delta))x^2$. Using this with $x = (\delta_1+\delta_2)-(\delta_1^2+\delta_2^2)/2$ and $\delta = (2R-1)/R^2$, which implies $\delta/(1-\delta) \le 6/R$ we find, after some calculation, that this term can be estimated by

$$\frac{2}{|1-\delta_1|^2+|1-\delta_2|^2} \le 1+(\delta_1+\delta_2)+2\delta_1\delta_2+\left(\frac{1}{2}+\frac{40}{R}\right)(\delta_1^2+\delta_2^2).$$

We now turn to the middle term on the right side of (22). Multiplying out the square, using $\mathrm{Re}(u_\lambda) = \lambda/2$, and making some simple estimates, we arrive at

$$\begin{aligned}|2(u-u_\lambda)-(\delta_1+\delta_2)(2u-u_\lambda)+2\delta_1\delta_2 u|^2 \\ &\le 4|u-u_\lambda|^2-4(\delta_1+\delta_2)(|u-u_\lambda|^2+u(u-\lambda/2)) \\ &\quad +2\delta_1\delta_2(|2u-u_\lambda|^2+4u(u-\lambda/2)) \\ &\quad +(\delta_1^2+\delta_2^2)(|2u-u_\lambda|^2+R^{-1}(9|u|^2+2|\lambda u|)).\end{aligned}$$

We now combine these estimates. In the error terms, we can control quadratic terms in u using

$$|u|^2 \le \frac{1}{1-\lambda^2/8}|u-u_\lambda|^2.$$

A straightforward calculation completes the proof. $\square$

In preparation for the proof of Lemma 4 we prove the following lemma. Recall that ω_1 and ω_2 are functions of u and $\mathbf{q}$. Explicitly,

$$\omega_i(u,\mathbf{q}) = \frac{u-q_i}{\sqrt{(u-q_1)^2+(u-q_2)^2}},$$

so that $\omega_i(u,a\mathbf{q}) = \omega_i(u/a,\mathbf{q})$. Also, with the notation of (8) we have

$$\int_{\mathbb{R}^2}(q_1-q_2)^2 d\nu(\mathbf{q}) = c_{11}+c_{22}-2c_{12} = c(1-\delta).$$

Lemma 8 *For $R \ge 2$ and $|u| \le aR$,*

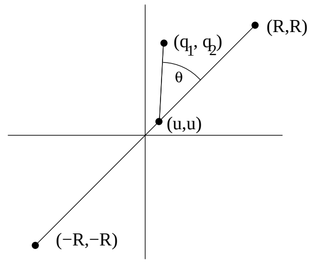
$$\int_{\mathbb{R}^2}\frac{|\omega_1+\omega_2|^2}{2}d\nu_a(\mathbf{q}) \le 1-\frac{c(1-\delta)}{20R^2}. \tag{23}$$

Proof: We begin with a scaling argument. The scaling properties of $\omega_i(u,\mathbf{q})$ and ν_a imply that bounding the left side of (23) for $|u| \le aR$ is equivalent to bounding

$$\int_{\mathbb{R}^2}\frac{|\omega_1+\omega_2|^2}{2}d\nu(\mathbf{q})$$

for $|u| \le R$.

Referring to the following diagram, we have $\omega_1 = -\cos(\theta + \pi/4)$ and $\omega_2 = -\sin(\theta + \pi/4)$.



Co-ordinates ω_1, ω_2 relative to (u, u).

Then $|\omega_1 + \omega_2|^2/2 = (1 + 2\omega_1\omega_2)/2 = (1 + \cos(2\theta))/2$. From this we see that the maximum occurs at an endpoint for θ, when $(u, u) = (R, R)$ or $(u, u) = (-R, -R)$. This leads to

$$\frac{|\omega_1 + \omega_2|^2}{2} \leq \frac{|\pm R - \bar{q}|^2}{|\pm R - \bar{q}|^2 + \tilde{q}^2} = 1 - \frac{\tilde{q}^2}{|R \pm \bar{q}|^2 + \tilde{q}^2},$$

where $\bar{q} = (q_1 + q_2)/2$ and $\tilde{q} = (q_1 - q_2)/2$. Since $|\bar{q}| \leq 1$ and $R \geq 2$ we have $|R \pm \bar{q}| \leq 2R$. This implies

$$\frac{|\omega_1 + \omega_2|^2}{2} \leq 1 - \frac{\tilde{q}^2}{4R^2 + 1} = 1 - \frac{(q_1 - q_2)^2}{20R^2}.$$

Integrating this formula completes the proof. □

We are now ready to give the proof of Lemma 4.

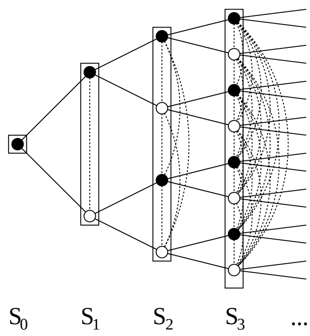
Proof of Lemma 4: The estimates of Lemma 6, Lemma 7 and the estimate $(1 + x)^{1+\alpha} \leq 1 + (1 + \alpha)x + \alpha(1 + \alpha)x^2$ for $x > -1$ can be used to show

$$\mu_{\mathbf{q},\lambda}(u)^{1+\alpha} \leq \begin{cases} \dfrac{|\omega_1 + \omega_2|^2}{2} + C_1 aR + C_2\alpha & \text{for } |u| \leq aR \\[2ex] 1 + \dfrac{1 + \alpha}{u} \langle \mathbf{l}, \mathbf{q} \rangle - \dfrac{1}{u^2} \langle \mathbf{q}, (Q - C_3/R - C_4\alpha)\mathbf{q} \rangle & \text{for } |u| \geq aR \end{cases}.$$

We now integrate this estimate with respect to ν_a. For $|u| \leq aR$, we use Lemma 8. When we integrate the estimate for $|u| \geq aR$, the linear term vanishes, thanks to (7). The quadratic term gives the estimate on the right side in (15). □

7. A mean field model

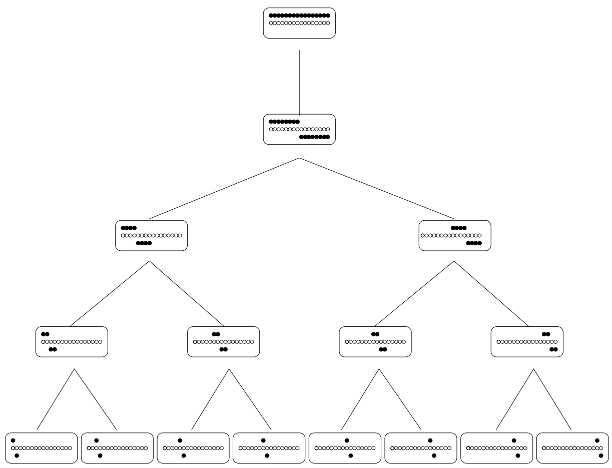
In this section we add a weighted complete graph to every sphere in the tree. Since the weights are chosen to make the total added weights the same in each sphere, this is a sort of mean field model. Pick a number $\gamma > 0$. Each added edge (dotted line in the diagram below) in the nth sphere S_n is given the weight $\gamma 2^{-n}$.



Rooted binary tree with mean-field edges insides spheres and transversely 2-periodic potential.

We call this graph the mean field binary tree. The spectrum of the free Laplacian on the mean field tree is the union of two intervals $[-2\sqrt{2}+\gamma, 2\sqrt{2}+\gamma] \cup [-2\sqrt{2}, 2\sqrt{2}]$ and is purely absolutely continuous. This can be seen by diagonalizing the Laplacian using a Haar basis, as in [AF]. The (normalized) Haar basis $\{e_0, e_1, \ldots, e_{2^n-1}\}$ for $\mathbb{C}^{2^n}$ is defined as follows. Let $(e_0)(j) = 2^{-n/2}, j = 1, \ldots, 2^n$. For $k = 0, 1, \ldots, n-1$ we set $(e_{2^k})(j) = 2^{-(n-k)/2}$ if $j = 1, \ldots, 2^{n-k-1}$, $(e_{2^k})(j) = -2^{-(n-k)/2}$ if $j = 2^{n-k-1}+1, \ldots, 2^{n-k}$, and 0 otherwise. Finally, we define the non-zero components of e_i for $2^k \le i < 2^{k+1}$ to be $(e_i)(j) = (e_{2^k})(j - i + 2^k)$.

Here is a diagram of the Haar basis for $\ell^2(S^n) = \mathbb{C}^{2^n}$ with $n = 3$. Each vector is normalized to make the basis orthonormal. This basis has a natural tree structure determined by the supports of the vectors. The highest level is the constant vector, and the lowest level consists of vectors with two non-zero entries of $\pm 2^{-1/2}$.



Haar basis for $\ell^2(S^3)$.

To simplify the calculations, we will consider this model when the transversely two-periodic potential is defined by the product of two independent Bernoulli distributions for q_1 and q_2,

$$\nu = \frac{1}{4}\big(\delta(q_1 - 1) + \delta(q_1 + 1)\big)\big(\delta(q_2 - 1) + \delta(q_2 + 1)\big).$$

Theorem 9 *Let $\nu_{(0)}$ be a probability measure of bounded support for the potential at the root and ν be the product of Bernoulli distributions defined above and let $H_{a,\gamma}$ be the random discrete Schrödinger operator on the mean field binary tree corresponding to the transversely two-periodic potential defined by the scaled distribution ν_a and weight γ. There exist $0 < \lambda_0, \lambda_1 < 2\sqrt{2}$ such that for sufficiently small a the spectral measure for H_a corresponding to δ_0 has purely absolutely continuous spectrum in $\{\lambda : |\lambda| \le \lambda_0, |\lambda - \gamma| \le \lambda_1\}$.*

In this theorem, the constant λ_0 has the same value as in the first part of the paper, while λ_1 can be taken to be any positive number less than $2\sqrt{2}$.

The forward Green's functions G_n are not diagonal. In the basic recursion formula (1) for the forward Green's functions on the mean field tree the matrices E_n and Q_n are unchanged from the binary tree, but the matrices D_n are now $2^{-n}\gamma$ times the Laplace operator for the complete graph on S_n. This Laplace operator is a $2^n \times 2^n$ matrix with each diagonal entry equal to zero and each off-diagonal entry equal to 1. Thus

$$D_n = \gamma(P - 2^{-n}I),$$

where P projects onto $2^{-n/2}[1, 1, \ldots, 1]^T$. Introduce the $d_n \times d_n$ matrix

$$U_n = E_n^T G_{n+1} E_n - D_n + \lambda = E_n^T G_{n+1} E_n - \gamma P + \lambda_n,$$

where

$$\lambda_n = \lambda + \gamma 2^{-n}. \tag{24}$$

Then the basic recursion formula reads

$$U_{n-1} = -E_{n-1}^T (U_n - Q_n)^{-1} E_{n-1} - \gamma P + \lambda_{n-1}.$$

The range of P is the span of the first vector in the Haar basis. Since the representation of a two-periodic potential in this basis is not too complicated, it is natural to change to this basis to simplify the problem.

Let V_n be the $2^n \times 2^n$ orthogonal change of basis matrix to the Haar basis, whose columns consist of the Haar basis vectors.

Lemma 10

(i) $V_n^T P V_n = \mathrm{diag}[1, 0, 0, \ldots]$.

(ii) $V_n^T E_n^T V_{n+1} = \sqrt{2}[I, \mathbf{0}]$.

(iii) Let $Q = \mathrm{diag}[q_1, q_2, q_1, q_2, \ldots]$ be a two-periodic potential. Setting $\bar{q} = (q_1 + q_2)/2$ and $\tilde{q} = (q_1 - q_2)/2$ we have

$$V_n^T Q V_n = \bar{q} I + \tilde{q} \begin{bmatrix} \mathbf{0} & V_{n-1}^T \\ V_{n-1} & \mathbf{0} \end{bmatrix}.$$

The proof of this lemma is a straightforward computation, which we omit. Now we write the matrix U_n in the Haar basis. Define

$$\tilde{U}_n = V_n^T U_n V_n.$$

In view of Lemma 10, the recursion formula for $\tilde{U}_n$ reads

$$\tilde{U}_{n-1} = -2[I, \mathbf{0}] \left(\tilde{U}_n - \bar{q} - \tilde{q} \begin{bmatrix} \mathbf{0} & V_{n-1}^T \\ V_{n-1} & \mathbf{0} \end{bmatrix} \right)^{-1} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} - \gamma \operatorname{diag}[1, 0, 0, \ldots] + \lambda_{n-1}, \tag{25}$$

where λ_n is given by (24). This recursion formula preserves matrices of the form $\operatorname{diag}[u_1, u_2, u_2, \ldots]$.

Lemma 11 *Suppose that $\tilde{U}_n = \operatorname{diag}[u_1, u_2, u_2, \ldots]$. Then $\tilde{U}_{n-1}$, defined by the recursion formula above, has the form*

$$\tilde{U}_{n-1} = \operatorname{diag}[\psi_{\mathbf{q},\lambda,\gamma,n-1}(u_1, u_2), \phi_{\mathbf{q},\lambda,n-1}(u_2), \phi_{\mathbf{q},\lambda,n-1}(u_2), \ldots],$$

where

$$\begin{aligned} \psi_{\mathbf{q},\lambda,\gamma,n}(u_1, u_2) &= -\frac{2}{u_1 - \bar{q} - \tilde{q}^2 (u_2 - \bar{q})^{-1}} + \lambda_n - \gamma, \\ \phi_{\mathbf{q},\lambda,n}(u_2) &= -\frac{2}{u_2 - \bar{q} - \tilde{q}^2 (u_2 - \bar{q})^{-1}} + \lambda_n, \end{aligned} \tag{26}$$

and λ_n is given by (24).

Proof: We have

$$\left(\tilde{U}_n - \bar{q} - \tilde{q} \begin{bmatrix} \mathbf{0} & V_{n-1}^T \\ V_{n-1} & \mathbf{0} \end{bmatrix} \right)^{-1} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix},$$

where

$$\begin{aligned} A &= \operatorname{diag}[u_1 - \bar{q}, u_2 - \bar{q}, u_2 - \bar{q}, \ldots], \\ B &= -\tilde{q} V_{n-1}, \\ C &= (u_2 - \bar{q}) I. \end{aligned}$$

The top left block of this inverse is given by Schur's formula $(A - B^T C^{-1} B)^{-1}$. Since $B^T C^{-1} B = \tilde{q}^2 (u_2 - \bar{q})^{-1} V_{n-1}^T V_{n-1} = \tilde{q}^2 (u_2 - \bar{q})^{-1} I$, the result is a diagonal matrix with $\big(u_1 - \bar{q} - \tilde{q}^2 (u_2 - \bar{q})^{-1}\big)^{-1}$ in the upper left corner and $\big(u_2 - \bar{q} - \tilde{q}^2 (u_2 - \bar{q})^{-1}\big)^{-1}$ in the other diagonal positions. The recursion formula picks out this block, multiplies by -2 and then adds $-\gamma \operatorname{diag}[1, 0, 0, \ldots] + \lambda_{n-1}$. This gives the formulas (26). □

The fact that the recursion formula for $\tilde{U}_n$ preserves diagonal matrices having the form $\operatorname{diag}[u_1, u_2, u_2, \ldots]$ means that $\tilde{U}_n$ must actually have this form. This follows from the limit formula for the forward Green's functions proved in [FHS1] which implies that these matrices will lie in any set that is preserved by the recursion flow. Thus, there are two random variables u_1 and u_2 for each sphere that describe the forward Green's function. For the nth sphere, they are distributed according to some joint measure $\rho_{a,\lambda,\gamma,n}$ for (u_1, u_2). Since the variables for adjacent spheres are related by (26) the recursion formula for these measures reads

$$\int_{\mathbb{H}\times\mathbb{H}} w(u_1, u_2) d\rho_{a,\lambda,\gamma,n}(u_1, u_2)$$
$$= \int_{\mathbb{H}\times\mathbb{H}} \int_{\mathbb{R}^2} w(\psi_{\mathbf{q},\lambda,\gamma,n}(u_1, u_2), \phi_{\mathbf{q},\lambda,n}(u_2))\, d\nu_a(\mathbf{q})\, d\rho_{a,\lambda,\gamma,n+1}(u_1, u_2).$$

Define the moments

$$M_{a,\alpha,\lambda,\gamma,n} = \int_{\mathbb{H}\times\mathbb{H}} \mathrm{cd}_{1,n}(u_1)^{1+\alpha} d\rho_{a,\lambda,\gamma,n}(u_1, u_2),$$

where

$$\mathrm{cd}_{1,n}(u_1) = \frac{|u_1 - u_{\lambda_n - \gamma}|^2}{\mathrm{Im}(u_1)}$$

and u_λ is the same fixed point as in (11). Our goal is to bound $M_{a,\alpha,\lambda,\gamma,0}$ for a and α small and λ and γ in some range. When $n = 0$ then $\tilde{U}_0 = U_0 = [u_1] = E_0^T G_1 E_0 + \lambda$. Since $G_0 = -(E_0^T G_1 E_0 + \lambda - q_0)^{-1}$ we can use the argument of Lemma 3 to prove the existence of absolutely continuous spectrum from such a bound.

Observe now that the recursion for u_2 is the same as the formula for u in the first part of the paper, except that λ is replaced by λ_n. Explicitly,

$$\phi_{\mathbf{q},\lambda_n}(u) = \phi_{\mathbf{q},\lambda,n}(u),$$

where the ϕ is given on the left by (10) and on the right by (26). We claim this implies that

$$M^{(2)}_{a,\alpha,\lambda,\gamma,n} = \int_{\mathbb{H}\times\mathbb{H}} \mathrm{cd}_{2,n}(u_2)^{1+\alpha} d\rho_{a,\lambda,\gamma,n}(u_1, u_2) \le C, \tag{27}$$

provided $|\lambda| < \lambda_0$. Here

$$\mathrm{cd}_{2,n}(u_2) = \frac{|u_2 - u_{\lambda_n}|_+^2}{\mathrm{Im}(u_2)}.$$

The function $|z|_+$ is equal to $|z|$ except near $z = 0$ where it has been modified to be bounded away from zero. This makes no difference to the growth properties, but will allow us to make a needed lower bound in the next section. For large n the bound (27) follows from the results in the first part of the paper (extended to distributions that vary from sphere to sphere) since the small perturbations $\gamma 2^{-n}$ of λ are easily absorbed in the proof. The result for large n suffices, since it is easy to iterate the bound (27) a finite number of steps. All that is required is an upper bound $\mu_{\mathbf{q},\lambda_n}(u) \le C$, for μ given by (14).

Similarly, it is enough to bound $M_{a,\alpha,\lambda,\gamma,n}$ for large n. We follow the same basic steps as before to begin the proof of such a bound. Let $\chi(u_1)$ be a cutoff with support where u_1 is in a neighbourhood of $\partial_\infty \mathbb{H}$. Then, one iteration gives

$$
\begin{aligned}
&M_{a,\alpha,\lambda,\gamma,n} \\
&\quad= \int_{\mathbb{H}\times\mathbb{H}} \mathrm{cd}_{1,n}(u_1)^{1+\alpha} d\rho_{a,\lambda,\gamma,n}(u_1,u_2) \\
&\quad\leq \int_{\mathbb{H}\times\mathbb{H}} \int_{\mathbb{R}^2} \mathrm{cd}_{1,n}(\psi_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2))^{1+\alpha} \chi(u_1) d\nu_a(\mathbf{q}) d\rho_{a,\lambda,\gamma,n+1}(u_1,u_2) + C \\
&\quad= \int_{\mathbb{H}\times\mathbb{H}} \int_{\mathbb{R}^2} \left(\mathrm{cd}_{1,n}(\psi_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2)) - C_1 \mathrm{cd}_{2,n}(u_2) + C_1 \mathrm{cd}_{2,n}(u_2)\right)^{1+\alpha} \\
&\qquad\quad \cdot \chi(u_1) d\nu_a(\mathbf{q}) d\rho_{a,\lambda,\gamma,n+1}(u_1,u_2) + C \\
&\quad\leq \int_{\mathbb{H}\times\mathbb{H}} \int_{\mathbb{R}^2} 2^\alpha \left[\mathrm{cd}_{1,n}(\psi_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2) - C_1 \mathrm{cd}_{2,n}(u_2)\right]_+^{1+\alpha} \chi(u_1) d\nu_a(\mathbf{q}) d\rho_{a,\lambda,\gamma,n+1}(u_1,u_2) \\
&\qquad\quad + 2^\alpha C_1^{1+\alpha} M^{(2)}_{a,\alpha,\lambda,\gamma,n} + C.
\end{aligned}
$$

The notation $[x]_+$ denotes $\max\{0,x\}$, not to be confused with $|z|_+$. Here we used the convexity of $x \mapsto x^{1+\alpha}$. The positive constant C_1 can be chosen as large as we please.

Now we define

$$
\mu_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2) = \frac{\mathrm{cd}_{1,n}(\psi_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2)) - C_1 \mathrm{cd}_{2,n}(u_2)}{\mathrm{cd}_{1,n+1}(u_1)}, \tag{28}
$$

and the averaged version

$$
\overline{\mu}_{a,\alpha,\lambda,\gamma,n}(u_1,u_2) = \int_{\mathbb{R}^2} \left[\mu_{\mathbf{q},\lambda,\gamma,n}\right]_+^{1+\alpha}(u_1,u_2)\, d\nu_a(\mathbf{q}).
$$

Then, provided $|\lambda| \leq \lambda_0$ so that $M^{(2)}_{a,\alpha,\lambda,\gamma,n}$ is bounded, we can rewrite the estimate above as

$$
M_{a,\alpha,\lambda,\gamma,n} \leq \int_{\mathbb{H}\times\mathbb{H}} 2^\alpha \overline{\mu}_{a,\alpha,\lambda,\gamma,n}(u_1,u_2)\, \chi(u_1)\, \mathrm{cd}_{1,n+1}(u_1)^{1+\alpha} d\rho_{a,\lambda,\gamma,n+1}(u_1,u_2) + C.
$$

A second iteration results in

$$
\begin{aligned}
M_{a,\alpha,\lambda,\gamma,n} \leq \int_{\mathbb{H}\times\mathbb{H}} \int_{\mathbb{R}^2} & 2^{2\alpha} \overline{\mu}_{a,\alpha,\lambda,\gamma,n}(\psi_{\mathbf{q},\lambda,\gamma,n+1}(u_1,u_2), \phi_{\mathbf{q},\lambda,n+1}(u_2)) \chi(\psi_{\mathbf{q},\lambda,\gamma,n+1}(u_1,u_2)) \\
& \cdot \left[\mu_{\mathbf{q},\lambda,\gamma,n+1}(u_1,u_2)\right]_+^{1+\alpha} \chi(u_1)\, d\nu_a(\mathbf{q})\, \mathrm{cd}_{1,n+2}^{1+\alpha}(u_1) d\rho_{a,\lambda,\gamma,n+2}(u_1,u_2) + C.
\end{aligned} \tag{29}
$$

Lemma 12 *There exist $0 < \lambda_0, \lambda_1 < 2\sqrt{2}$ such that for $|\lambda| \leq \lambda_0$, $|\lambda - \gamma| \leq \lambda_1$, a, α sufficiently small, n sufficiently large and χ supported sufficiently near $\partial_\infty \mathbb{H}$, there is $\epsilon > 0$ such that*

$$
\begin{aligned}
\int_{\mathbb{R}^2} & 2^{2\alpha} \overline{\mu}_{a,\alpha,\lambda,\gamma,n}(\psi_{\mathbf{q},\lambda,\gamma,n+1}(u_1,u_2), \phi_{\mathbf{q},\lambda,n+1}(u_2)) \chi(\psi_{\mathbf{q},\lambda,\gamma,n+1}(u_1,u_2)) \\
& \cdot \left[\mu_{\mathbf{q},\lambda,\gamma,n+1}(u_1,u_2)\right]_+^{1+\alpha} \chi(u_1) d\nu_a(\mathbf{q}) \leq 1 - \epsilon.
\end{aligned}
$$

This lemma, proved below, implies the main result for the mean field model.

Proof of Theorem 9: Inserting the estimate of Lemma 12 into (29) gives

$$M_{a,\alpha,\lambda,\gamma,n} \leq (1-\epsilon)M_{a,\alpha,\lambda,\gamma,n+2} + C$$

for n large. This is the same estimate as (21) so we can follow the argument given there to bound $M_{a,\alpha,\lambda,\gamma,n}$ for n large. As noted above, this is sufficient to prove the theorem. □

8. Proof of Lemma 12

The function $\mu_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2)$ is the rational function given by

$$\mu_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2) = \frac{|(u_2-\bar{q})u_{\lambda_n-\gamma} - (u_1-\bar{q})(u_2-\bar{q}) + \tilde{q}^2|^2\,\mathrm{Im}(u_1)}{\big(|u_2-\bar{q}|^2\,\mathrm{Im}(u_1) + \tilde{q}^2\,\mathrm{Im}(u_2)\big)|u_1-u_{\lambda_{n+1}-\gamma}|^2} - C_1\frac{\mathrm{Im}(u_1)|u_2-u_{\lambda_n}|_+^2}{\mathrm{Im}(u_2)|u_1-u_{\lambda_{n+1}-\gamma}|^2}.$$

For $|\lambda-\gamma| \leq \lambda_1 < 2\sqrt{2}$, the fixed point $u_{\lambda_n-\gamma}$ lies in the upper half plane for n sufficiently large, and is bounded away from $\partial_\infty\mathbb{H}$. The function $\mu_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2)$ always appears with a cutoff function $\chi(u_1)$ that ensures that u_1 is in a neigbourhood of $\partial_\infty\mathbb{H}$ and thus that, for n sufficiently large, $|u_1-u_{\lambda_{n+1}-\gamma}|$ is bounded below by a positive constant. The variable u_2 can range over all of $\mathbb{H}$.

Introduce polar co-ordinates r, ω_1 and ω_2 for $\mathrm{Im}(u_1)$ and $\mathrm{Im}(u_2)$ as

$$\mathrm{Im}(u_1) = r\omega_1, \quad \mathrm{Im}(u_2) = r\omega_2, \quad \omega_1^2+\omega_2^2 = 1.$$

Then

$$\mu_{\mathbf{q},\lambda,\gamma,n}(u_1,u_2) = \frac{|(u_2-\bar{q})u_{\lambda_n-\gamma} - (u_1-\bar{q})(u_2-\bar{q}) + \tilde{q}^2|^2\omega_1}{\big(|u_2-\bar{q}|^2\omega_1 + \tilde{q}^2\omega_2\big)|u_1-u_{\lambda_{n+1}-\gamma}|^2} - C_1\frac{\omega_1|u_2-u_{\lambda_n}|_+^2}{\omega_2|u_1-u_{\lambda_{n+1}-\gamma}|^2}.$$

With a Bernoulli distribution, the potential takes on four possible values $(\pm a, \pm a)$. The corresponding values of μ are as follows.

$$\mu^{++}_{a,\lambda,\gamma,n}(u_1,u_2) = \frac{|u_1-u_{\lambda_n-\gamma}-a|^2}{|u_1-u_{\lambda_{n+1}-\gamma}|^2} - C_1\frac{\omega_1|u_2-u_{\lambda_n}|_+^2}{\omega_2|u_1-u_{\lambda_{n+1}-\gamma}|^2}.$$

The formula for μ^{--} is identical, except that $-a$ is replaced with a. For the other two values, we have $\mu^{+-} = \mu^{-+}$, with

$$\mu^{+-}_{a,\lambda,\gamma,n}(u_1,u_2) = \frac{|u_2(u_1-u_{\lambda_n-\gamma}) - a^2|^2\omega_1}{\big(|u_2|^2\omega_1 + a^2\omega_2\big)|u_1-u_{\lambda_{n+1}-\gamma}|^2} - C_1\frac{\omega_1|u_2-u_{\lambda_n}|_+^2}{\omega_2|u_1-u_{\lambda_{n+1}-\gamma}|^2}.$$

Lemma 13 *For u_1 in a neigbourhood of $\partial_\infty \mathbb{H}$ and a bounded,*

$$\mu^{++}_{a,\lambda,\gamma,n}(u_1,u_2) \le 1 + C\frac{a+2^{-n}}{|u_1 - u_{\lambda_{n+1}-\gamma}|}.$$

Proof: Dropping the second term we have

$$\mu^{++}_{a,\lambda,\gamma,n}(u_1,u_2) \le \left|1 + \frac{u_{\lambda_{n+1}-\gamma} - u_{\lambda_n-\gamma} - a}{u_1 - u_{\lambda_{n+1}-\gamma}}\right|^2.$$

Expanding the square, using that $|u_{\lambda_n-\gamma} - u_{\lambda_{n+1}-\gamma}| \le C2^{-n}$ and that $|(a+C2^{-n})/(u_1 - u_{\lambda_{n+1}-\gamma})|$ is bounded, since a is bounded and u_1 is bounded away from $u_{\lambda_{n+1}-\gamma}$ near $\partial_\infty \mathbb{H}$ completes the proof. $\square$

The following lemma is the most involved estimate in this section.

Lemma 14 *Suppose that u_1 lies in a sufficiently small neighbourhood of infinity. Then for C_1 and n sufficiently large and a sufficiently small, there exists a positive constant C such that*

$$\mu^{+-}_{a,\lambda,\gamma,n}(u_1,u_2) \le 1 - \frac{C\sqrt{C_1}(a-2^{-n})}{|u_1 - u_{\lambda_{n+1}-\gamma}|}. \tag{30}$$

Proof: To simplify the appearance of the formulas, we introduce the notation

$$A_n = u_1 - u_{\lambda_n-\gamma}, \quad B_n = u_2 - u_{\lambda_n}.$$

We begin by establishing the inequality

$$\mu^{+-}_{a,\lambda,\gamma,n}(u_1,u_2) \le \frac{\left[|u_2A_n - a^2| - a\sqrt{C_1}|B_n|_+\right]_+^2}{|u_2|^2|A_{n+1}|^2}. \tag{31}$$

Let $x = \omega_2/\omega_1 \in [0,\infty]$. We must maximize

$$\mu^{+-}_{a,\lambda,\gamma,n}(u_1,u_2) = \frac{|u_2A_n - a^2|^2}{(|u_2|^2 + xa^2)|A_{n+1}|^2} - C_1\frac{|B_n|_+^2}{x|A_{n+1}|^2}$$

over x. We will assume without loss that $a > 0$. Differentiating with respect to x we obtain the following equation for the critical point:

$$\frac{|u_2A_n - a^2|^2a^2}{(|u_2|^2 + xa^2)^2} = \frac{C_1|B_n|_+^2}{x^2},$$

or

$$|u_2A_n - a^2|ax = \pm\sqrt{C_1}|B_n|_+(|u_2|^2 + xa^2).$$

Since x is non-negative we must choose $\pm = +$. This results in the critical point

$$x = \frac{\sqrt{C_1}|B_n|_+|u_2|^2}{a(|u_2A_n - a^2| - a\sqrt{C_1}|B_n|_+)}.$$

The critical point will lie in $[0, \infty]$ provided

$$|u_2 A_n - a^2| \geq a\sqrt{C_1}|B_n|_+, \tag{32}$$

in which case a calculation shows that the critical value is

$$\frac{\left(|u_2 A_n - a^2| - a\sqrt{C_1}|B_n|_+\right)^2}{|u_2|^2 |A_{n+1}|^2}. \tag{33}$$

At the endpoint $x = 0$ we find that μ^{+-} tends to $-\infty$ while the limit as $x \to \infty$ is 0. This implies that when (32) holds, then the maximum occurs at the critical value, and otherwise the maximum is 0. This proves (31).

Now we can proceed with the proof of estimate (30). We may assume that (32) holds, because otherwise μ^{+-} is zero and the desired estimate is true. This implies that for some $\epsilon > 0$ (e.g., $\epsilon = \frac{a}{\sqrt{C_1}|B_n|_+}$)

$$|u_2 A_n| \geq a(1-\epsilon)\sqrt{C_1}|B_n|_+.$$

Here we use that $|B_n|_+ \geq C$. Thus we may assume

$$\frac{a\sqrt{C_1}|B_n|_+}{|u_2 A_n|} \leq 1 + 2\epsilon \tag{34}$$

provided $0 < \epsilon < 1/2$ and use this in estimating (33). Expanding the square in (33) we end up with an estimate for μ^{+-} given by

$$\begin{aligned}\mu^{+-}_{a,\lambda,\gamma,n}(u_1, u_2) \leq{}& \frac{|A_n|^2}{|A_{n+1}|^2} \\ &+ \frac{a|B_n|_+}{|u_2||A_{n+1}|}\left(\frac{2a|A_n|}{|A_{n+1}||B_n|_+} + \frac{a^3}{|u_2||A_{n+1}||B_n|_+} - \frac{2\sqrt{C_1}|A_n|}{|A_{n+1}|} + \frac{2\sqrt{C_1}a^2}{|u_2||A_{n+1}|} + \frac{C_1 a|B_n|_+}{|u_2||A_{n+1}|}\right).\end{aligned}$$

Now we may use (34), $|B_n|_+ \geq C$ and $|A_n|/|A_{n+1}| \leq 1 + C2^{-n}/|A_{n+1}|$ to arrive at the estimate

$$\mu^{+-}_{a,\lambda,\gamma,n}(u_1, u_2) \leq 1 + C2^{-n}/|A_{n+1}| - \frac{a|B_n|_+}{|u_2||A_{n+1}|}((1-2\epsilon)\sqrt{C_1} - Ca).$$

Finally, the bound $|B_n|_+/|u_2| \geq C$ completes the proof. □

Proof of Lemma 12: With the Bernoulli distribution, the average defining $\overline{\mu}$ has four terms, so, dropping the subscripts and using the estimates from this section we have

$$\begin{aligned}\overline{\mu}(u_1, u_2) &= \frac{1}{4}\left([\mu^{++}]_+^{1+\alpha} + [\mu^{--}]_+^{1+\alpha} + [\mu^{+-}]_+^{1+\alpha} + [\mu^{-+}]_+^{1+\alpha}\right) \\ &\leq \frac{1}{2}\left(\left[1 + C\frac{a + 2^{-n}}{|u_1 - u_{\lambda_{n-1}-\gamma}|}\right]^{1+\alpha} + \left[1 - C\sqrt{C_1}\frac{a - 2^{-n}}{|u_1 - u_{\lambda_{n-1}-\gamma}|}\right]^{1+\alpha}\right)\end{aligned}$$

For a small and n large, both terms inside the square brackets are a small perturbation of 1. But since we are free to take C_1 large, we may assume that the relative size of the term with the good (negative) sign is much larger. This leads to the estimate

$$\overline{\mu}(u_1, u_2) \leq 1 - C\sqrt{C_1}\frac{a - 2^{-n}}{|u_1 - u_{\lambda_{n-1}-\gamma}|} + C(a + 2^{-n})^2$$

for a, α small and n, C_1 large.

To prove the lemma we must estimate the expression (again dropping most subscripts)

$$\frac{1}{4}\sum_{\mathbf{q}\in(\pm a,\pm a)} 2^{2\alpha}\overline{\mu}(\psi_{\mathbf{q}}(u_1,u_2),\phi_{\mathbf{q}}(u_1,u_2))\,\chi(\psi_{\mathbf{q}}(u_1,u_2))\,[\mu_{\mathbf{q}}(u_1,u_2)]_+^{1+\alpha}\,\chi(u_1)$$

When $|u_1| \le C$ we can estimate $\overline{\mu}$ by $1 + C(a + 2^{-n})^2$ and pull it out of the sum. What results is another copy of $\overline{\mu}$ evaluated at bounded u_1. This can be estimated by $1 - \epsilon$. Since for small α the quantity $2^{2\alpha}$ is close to 1, we end up with the desired bound of $1 - \epsilon$ for a, α small and n, C_1 large.

For u_1 near infinity we estimate the occurances of μ in the sum by the bound for μ^{++} which is slightly greater than one. Then we just need to guarantee that one of the $\overline{\mu}$ terms will be evaluated with $\psi_{\mathbf{q}}(u_1, u_2)$ bounded. This happens when $\mathbf{q} = (a, a)$ since in this case $\psi_{\mathbf{q}}(u_1, u_2) = -2/(u_1 - \bar{q}) + \lambda_n - \gamma$ indepedendently of u_2. $\square$

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