Liouville's theorem in the radially symmetric case

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Abstract

We present a very short proof of Liouville's theorem for solutions to a non-uniformly elliptic radially symmetric equation. The proof uses the Ricatti equation satisfied by the Dirichlet to Neumann map.

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Introduction

The classical version of Liouville's theorem asserts that if a harmonic function defined on all of Euclidean space is bounded, it must be constant. This fundamental result has been generalized in many directions, and its study for manifolds is a large field of research. Several years ago, in connection with their work on the De Giorgi conjecture, Ghoussoub and Gui [GG] raised the question of whether Liouville's theorem holds in Euclidean space for solutions of non uniformly elliptic equations of the form

$$\nabla \cdot \sigma^2 \nabla \varphi = 0$$

in place of harmonic functions. Here $\sigma(x)^2$ is bounded and positive, but need not be bounded away from zero. It was known from the work of Berestycki, Caffarelli and Nirenberg [BCN] that in two dimensions this version of Liouville's theorem holds. However Ghoussoub and Gui [GG] showed that that in dimensions 7 and higher it does not. The remaining cases, dimensions 3 through 6 were settled, also in the negative, by Barlow [B].

The situation is quite different if σ is assumed to be radially symmetric. In this case Liouville's theorem does hold in any dimension. This is known from work of Losev [L] (see also [LM], which contains further references). Of course, the radially symmetric case is much simpler, since separation of variables reduces the problem to question about ODE's. Specifically, any solution φ is a linear combination of solutions of the form $\varphi(x) = u(|x|)Y(x/|x|)$, where Y is an eigenfunction of the spherical Laplacian with eigenvalue μ^2, and u satisfies the ODE (1) below.

This paper is just a small remark about this simple case. We show that the Ricatti equation for the Dirichlet to Neumann map leads to a very short proof. Of course, in our setting the Dirichlet to Neuman map is just a function, namely the function f defined by (3) below. But it is interesting to note that the Dirichlet to Neumann map satisfies an operator version of the same Ricatti equation (4) that f satisfies, even when σ is not radially symmetric. Perhaps this can be used to give a proof of Liouville's theorem for perturbations of radially symmetric σ. However, we did not see a simple way of doing this.

Growth of solutions

Theorem 1 *Suppose that u solves the radial Laplace equation*

$$(\sigma^2(r)u'(r))' + \frac{(n-1)\sigma^2(r)}{r}u'(r) - \frac{\mu^2\sigma^2(r)}{r^2}u(r) = 0 \tag{1}$$

for $r \in (0,\infty)$. Assume $n \geq 3$ and $\sigma^2(r) \in C^1([0,\infty))$ with $0 < \sigma^2(r) \leq 1$. Define $\beta(\mu)$ to be

$$\beta(\mu) = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu^2}$$

If u is bounded by

$$|u(r)| \leq C(1 + r^\beta) \tag{2}$$

then $\beta \geq \beta(\mu)$.

Remark: This theorem gives a minimal growth rate, depending on μ, for a solution u to the radial Laplace equation. It implies Liouville's theorem, since a bounded solution ($\beta = 0$) is only possible when $\beta(\mu) = 0$. In this case $\mu = 0$, and $u(r) = C$ is the only solution bounded at the origin.

Remark: When $\sigma(r) = 1$ the solution bounded at the origin is $u(r) = r^{\beta(\mu)}$. Thus the value of $\beta(\mu)$ is optimal.

Proof: Suppose that u is a solution satisfying (2). We must show that $\beta \geq \beta(\mu)$.

Define

$$f(r) = \frac{r^{n-1}\sigma^2(r)u'(r)}{u(r)} \tag{3}$$

Then f satisfies the Ricatti type equation

$$f'(r) = r^{n-3}\mu^2\sigma^2(r) - \frac{1}{r^{n-1}\sigma^2(r)}f^2(r) \tag{4}$$

We begin by showing that f is well defined and positive. This follows from the ODE version of Calderon's identity, namely,

$$\sigma^2(r)u'(r)u(r)r^{n-1} = \int_0^r \sigma^2(s)\Big(u'(s)^2 + \frac{\mu^2}{s^2}u^2(s)\Big)s^{n-1}ds > 0 \tag{5}$$

To prove this we first integrate by parts to obtain

$$\begin{aligned} 0 &< \int_a^r \sigma^2(s)\Big(u'(s)^2 + \frac{\mu^2}{s^2}u^2(s)\Big)s^{n-1}ds \\ &= \sigma^2(s)u'(s)u(s)s^{n-1}\Big|_a^r - \int_a^r u(s)\Big\{\frac{d}{ds}\Big(\sigma^2u's^{n-1}\Big) - \frac{\mu^2}{s^2}u^2(s)s^{n-1}\Big\}ds \\ &= \sigma^2(s)u'(s)u(s)s^{n-1}\Big|_a^r \end{aligned} \tag{6}$$

Then (5) follows from

$$\lim_{a\to 0}\sigma^2(a)u'(a)u(a)a^{n-1} = 0 \tag{7}$$

To see this, notice that the equation implies that $(\sigma^2r^{n-1}u')' = \mu^2\sigma^2r^{n-3}u$, so that the bounds on σ^2 and u imply that near zero,

$$0 \le (\sigma^2r^{n-1}u')' \le Cr^{n-3}$$

Integrating from 0 to a and letting a tend to zero gives (7) and thus (5). This shows that neither u nor u' can vanish, and that f is well defined, does not blow up, and is positive. Since u cannot change sign, so we may as well assume that $u > 0$.

The idea of the proof is to estimate the quantity

$$Q(r) = \int_1^r u'(x)/u(x) + \epsilon f'(x)/f(x)dx \tag{8}$$

from above and below for large r.

We begin with the upper bound. Performing the integral in (8) yields

$$Q(r) = \ln(u(r)) + \epsilon\ln(f(r)) + C$$

Here C denotes a generic constant that may depend on μ but not r. Dropping the negative second term in (4) and recalling that $\sigma^2(r) \le 1$ yields

$$f'(r) \le r^{n-3}\mu^2\sigma^2(r) \le r^{n-3}\mu^2$$

Integrating this yields

$$f(r) \leq \frac{\mu^2}{n-2} r^{n-2} + C.$$

This estimate together with our growth assumptions on u imply

$$Q(r) \leq \beta \ln(r) + \epsilon(n-2) \ln(r) + C$$

Now we turn to the lower bound. Using the expression given by (4) for f' we obtain

$$\begin{aligned} u'(x)/u(x) + \epsilon f'(x)/f(x) &= \frac{f(r)}{r^{n-1}\sigma^2(r)} + \epsilon \left(\frac{r^{n-3}\mu^2\sigma^2(r)}{f(r)} - \frac{f}{r^{n-1}\sigma^2(r)} \right) \\ &\geq (1-\epsilon) \frac{f(r)}{r^{n-1}\sigma^2(r)} + \epsilon \frac{r^{n-3}\mu^2\sigma^2(r)}{f(r)} \end{aligned}$$

Now assume that $0 \leq \epsilon \leq 1$. Since for positive numbers a, b and c we have

$$ac + b/c \geq 2\sqrt{ab}$$

we find

$$u'(x)/u(x) + \epsilon f'(x)/f(x) \geq 2\mu\sqrt{\epsilon(1-\epsilon)}/r.$$

This implies

$$Q(r) \geq 2\mu\sqrt{\epsilon(1-\epsilon)} \ln(r) + C$$

Comparing the upper and lower bounds for $Q(r)$ for large r yields

$$\beta \geq 2\mu\sqrt{\epsilon(1-\epsilon)} - \epsilon(n-2)$$

Since this holds for every $\epsilon \in [0,1]$, the theorem now follows from the computation

$$\max_{\epsilon\in[0,1]} 2\mu\sqrt{\epsilon(1-\epsilon)} - \epsilon(n-2) = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu^2} = \beta(\mu)$$

□

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