Journées *Équations aux dérivées partielles*
XXX, ?? mai–?? juin ????
GDR 1151 (CNRS)

On the distribution of resonances for some asymptotically hyperbolic manifolds

R. G. Froese and P. D. Hislop

Abstract

We establish a sharp upper bound for the resonance counting function for a class of asymptotically hyperbolic manifolds in arbitrary dimension, including convex, cocompact hyperbolic manifolds in two dimensions. The proof is based on the construction of a suitable paramatrix for the absolute S-matrix that is unitary for real values of the energy. This paramatrix is the S-matrix for a model Laplacian corresponding to a separable metric near infinity. The proof of the upper bound on the resonance counting function requires estimates on the growth of the relative scattering phase, and singular values of a family of integral operators.

1. Introduction

We sketch an outline of the proof of a sharp upper bound on the counting function for resonances (or scattering poles) for the Laplace operator on asymptotically hyperbolic manifolds of dimension greater than or equal to two. This class of manifolds includes infinite volume, convex, co-compact hyperbolic manifolds in dimension two (cf. [8, 14]). Our main technical results are two: A bound on the singular values of a regularization of the scattering operator, and an estimate on the growth of a relative scattering phase. The key observation is that a parametrix for the absolute S-matrix can be constructed which is unitary on the critical line. This construction uses the fact that the manifold admits a product structure and a boundary-defining function in a neighborhood of infinity. Our main theorem is the following.

Theorem 1.1 *Let X be an n-dimensional asymptotically hyperbolic manifold satisfying Hypothesis 1 given below. Let $\nu(r)$ denote the number of scattering poles contained in the semicircle of radius $r > 0$ centered at the origin. Then, for each $n \geq 2$, there exists a constant $0 < C_n < \infty$, such that we have the upper bound*

$$\nu(r) \leq C_n(r^n + 1).$$

Research partially supported by NSERC and NSF grant 9707049.

It is know from explicitly computable examples of hyperbolic manifolds that this upper bound is sharp. There have been many results on the distribution of scattering poles for hyperbolic manifolds. Guillopé and Zworski [9] proved an upper bound of the form $\nu(r) \leq C(r^{1+n} + 1)$, for complete manifolds of constant negative curvature at infinity in dimension $n \geq 3$. Using the Selberg zeta function associated with the convex, co-compact hyperbolic manifold, Patterson and Perry [18] proved the sharp upper bound $\nu(r) \leq C_n(r^n + 1)$, for arbitrary even $n \geq 2$. Using some results of Bunke and Olbrich [1], Patterson and Perry were able to extend their results for odd n odd. Quite recently, Perry [22] proved a Poisson formula for convex, cocompact hyperbolic manifolds used this to prove lower bounds of the same type on the resonance counting function. The methods of the Selberg zeta function appear to be very rigid. Our work is an attempt to use the S-matrix directly in order to prove these bounds. This allows us to include perturbations of the metric having rapid decay, as will be described below. Unfortunately, our present work cannot be applied directly to convex, co-compact hyperbolic manifolds of dimension $n \geq 3$. Our methods do prove, however, the following result. For any two constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$, let $\nu_{\epsilon_1,\epsilon_2}(r)$ be the number of scattering resonances in the sector $\pi + \epsilon_1 < \arg z < -\epsilon_2$ in the lower-half complex plane. Then, for any asymptotically hyperbolic manifold, there exists a constant $C_{n,\epsilon_1,\epsilon_2} > 0$, so that $\nu_{\epsilon_1,\epsilon_2}(r) \leq C_{n,\epsilon_1,\epsilon_2}(r^n + 1)$. We do not presently have enough control on the parametrix used in the proof of this estimate in order to extend the results to the full half-disk.

The case of dimension $n = 2$ is special. Guillopé and Zworski [10, 11] proved upper and lower bounds on the resonance counting function $\nu(r)$, and very precise asymptotics on the relative scattering phase. This analysis is facilitated by the fact that in two dimensions, the metric near an end at infinity is an exact product metric. In [11], the Guillopé and Zworski prove that there exists finite, nonzero constants c and C such that $cr^2 \leq \nu(r) \leq Cr^2$. Refining this analysis, Zworski [26] proved that the density of scattering poles in a subconic neighborhood near the real axis is controlled by the Hausdorff dimension of the limit set of the group of hyperbolic isometries Γ.

Upper bounds and asymptotics for scattering resonances have been the topics of many recent papers. We refer to the review article of Zworski [27] for an account of much of this work.

2. Preliminaries and Outline of the Proof

It is well-known that the absolute S-matrix $S(k)$ for an asymptotically hyperbolic manifold X is a pseudodifferential operator of complex order $-2ik$, $k \in C$ [13]. The meromorphic continuation of the S-matrix in the hyperbolic case has been proven in several works, including [1, 15, 16, 19, 6]. An interesting feature of the absolute S-matrix for asymptotically hyperbolic manifolds is that it is not of the form $S(k) = 1 + T(k)$, for some operator $T(k)$ in the trace class, as in the case of Schrödinger operators with potentials that decay as $|x|^{-(n+\epsilon)}$ near infinity. We can, however, construct a paramatrix $S_0(k)$ so that $S_0(k)^{-1}S(k) = 1 + T(k)$, with $T(k)$ trace class. Furthermore, the operator $S_0(k)$ extends meromorphically to the entire complex plane with computable poles. Such an operator is constructed in [6] (and, implicitly, in [1]) for a general class of geometrically finite, infinite volume hyperbolic manifolds, including cusps. The difficulty with that paramatrix is that $S_0(k)$ is not unitary on the $\Im k = 0$ axis. This parametrix, however, can be used to obtain the estimates on the resonance counting function for a sector mentioned in the introduction. In this note, we sketch the construction of a paramatrix for $S(k)$ that is unitary on the $\Im k = 0$ axis, and satisfies the same conditions mentioned above. Once this paramatrix is constructed, the proof of the upper bound on the resonance counting function will follow from Lemma 2.1, as explained below. Because the S-matrix satisfies the functional relation $S(k)S(-k) = 0$, for $k \in \mathbb{C}$, the poles of $S(k)$ for $\Im k < 0$ correspond to the zeros of $S(k)$ for $\Im k > 0$. It will be more convenient to count the number of zeros of $S(k)$ for $\Im k > 0$.

With reference to the relation $S_0(k)^{-1}S(k) = 1 + T(k)$, let $T(k)$ be an operator-valued function analytic on the half-space $\Im k > 0$, continuous onto the real axis, and in the trace class for $\Im k \geq 0$. We further assume that $(1 + T(k))$ is unitary for $\Im k = 0$, and that $T(0) = 0$. Let $F(k) \equiv \det\,(1+T(k))$. This function is analytic for $\Im k > 0$ and continuous onto $\Im k = 0$. For any $r > 0$, we denote by $\nu(r)$ the number of zeros z_i of $F(k)$ satisfying $\Im z_i > 0$ and $|z_i| < r$, including multiplicity. We define an averaged quantity $N(r)$ by

$$N(r) = \int_0^r \frac{\nu(t)}{t}\, dt. \tag{1}$$

Since $|F(k)| = 1$, for $\Im k = 0$, we define the *scattering phase* $s(k)$, for $k \in \mathbb{R}$, by $s(k) \equiv (-i/2\pi) \log F(k)$. We will see that this is an analogue of the scattering phase in the usual Schrödinger operator setting.

Lemma 2.1 *Let $F(k) = det\,(1 + T(k))$ be as defined above. Then, we have*

$$N(r) \leq \frac{1}{2\pi} \int_0^r t^{-1} |s(t) - s(-t)| dt + \frac{1}{2\pi} \int_0^\pi \log\, |F(re^{i\theta})|\, d\theta. \tag{2}$$

The proof of this simple lemma is given in [3]. Let us note that we can recover an upper bound on $\nu(r)$ from one on $N(r)$ by using the positivity and monotonicity of $\nu(r)$. We have

$$r\nu(r) \leq \int_r^{2r} \nu(t) dt \leq 2r \int_r^{2r} \frac{\nu(t)}{t} dt, \tag{3}$$

from which it follows that $\nu(r) \leq 2(N(2r) - N(r))$. Consequently, it suffices to obtain an upper bound on $N(r)$. This requires an upper bound on the scattering phase $s(t)$ of the form $|t|^n$, and an estimate on the growth of $|F(re^{i\theta})|$, as $r \to \infty$, of the form $e^{C(r^n+1)}$.

In the next section, we review the structure of the S-matrix associated to X, and show how to construct the S-matrix $S_0(k)$ using the geometry of the manifold X. We will denote by P the Laplace-Beltrami operator on X with an asymptotically hyperbolic metric g. The spectral theory of P is well-understood [6]. The approximate S-matrix $S_0(k)$ is associated with an approximate operator P_0 defined on the end of X obtained by removing a compact set U. The scattering phase $s(t)$ is the usual scattering phase associated with the pair (P_0, P). In section 4, we will estimate the relative scattering phase defined by $s(k) = (-i/2\pi) \log \{\det S_0(k)^{-1} S(k)\}$, for $k \in I\!R$. This estimate is obtained by following the method of Robert [23, 24] (see also Christiansen [2]). The method of Robert requires the construction of a pseudodifferential operator A_0 which satisfies $[P_0, A_0] = P_0$, near infinity, up to a smoothing operator. The operator A_0 has a symbol a that is a global escape function for the symbol of the approximate operator P_0. The idea behind the construction of this symbol is given in section 5. In section 6, we estimate the singular values of certain integral operators. These estimates are the key to estimating $F(re^{i\theta})$. The estimate on the growth of the scattering phase and on the singular values are the two main steps in the proof of Theorem 1.1. The details and proofs will be given in [4].

3. The Geometry of Asymptotically Hyperbolic Manifolds

We introduce the class of manifolds for which our methods work. The key assumption is that the metric can be well-approximated by a product metric near infinity. This model metric will provide a model operator P_0 for which we can apply the usual methods of scattering theory to the pair (P_0, P).

3.1. Description of the Metric

Let X be a precompact Riemannian manifold with metric g. We assume that the boundary ∂X of X is a nonempty, smooth, compact manifold, and such that the closure $\overline{X}$ is compact. A *boundary-defining function* for $\overline{X}$ is a real-valued function x defined on a neighborhood $X \backslash U$, for a compact set $U \subset X$, of the boundary ∂X of X. A boundary-defining function satisfies $\partial X = \{p \in \overline{X} \mid x(p) = 0\}$, and $dx | \partial X \neq 0$.

A manifold X of the above type is called *asymptotically hyperbolic* if for some compact $U \subset X$, the neighborhood of ∂X given by $X \backslash U$, admits a product decomposition $X \sim \partial X \times (0, \epsilon)$, for some $\epsilon > 0$, and that there exists a boundary-defining function $x : X \backslash U \rightarrow I\!R$, satisfying $\| dx \| \, | \partial X = 1$, so that metric g on $X \backslash U$ has the form

$$\frac{dx^2 + h(x, dx, y, dy)}{x^2}. \tag{4}$$

Furthermore, the function h has the property that $h | \partial X$ is a smooth metric on ∂X. It was observed in the article of Mazzeo and Melrose [16] that for metrics of this form, the sectional curvature computed along a smooth geodesic near ∂X, and running to the boundary point $p \in \partial X$, approaches -1. We will write (x, y) for coordinates on $X \backslash U$ near ∂X, with y denoting local coordinates on the compact manifold ∂X.

It is well-known, and proved explicitly in [13], that coordinates can chosen so that $h(x, y, dx, dy)$ is independent of dx. We will assume that this has been done. The form of the metric g near ∂X suggests a model for the end of X near ∂X given by $X \backslash U$. There, we define a model metric g_0 as follows. Let $h_0(y, dy) \equiv h(0, y, dy)$

be the metric on ∂X associated with h. We define the metric g_0 on $X \backslash U$ by

$$g_0 = \frac{dx^2 + h_0(y, dy)}{x^2}. \tag{5}$$

The model space for the manifold (X, g) is the manifold $(X \backslash U, g_0)$. Note that the model metric is separable, and hence we will be able to do computations involving g_0 explicitly. We need to make one crucial assumption on the behavior of h as $x \to 0$.

Hypothesis 1. *In any local coordinate chart for ∂X, the difference*

$$h(x, y, dy) - h_0(y, dy) = \mathcal{O}(x^{n-1+\epsilon}), \tag{6}$$

for some $\epsilon > 0$, and uniformly with respect to $y \in \partial X$.

This hypothesis is satisfied, for example, by a class of perturbations of convex, cocompact hyperbolic manifolds in two dimensions.

3.2. The Laplace-Beltrami and Model Operators

The Laplace-Beltrami operator P on $L^2(X, dV_g)$, where dV_g is the volume form associated with the metric g given in (4), is constructed from the asymptotically hyperbolic metric g on X (we can actually allow more general P). In the coordinates (x, y) near ∂X, the operator P has the form

$$P = -x^n (\det h)^{-1/2} \frac{\partial}{\partial x} x^{2-n} (\det h)^{1/2} \frac{\partial}{\partial x} + x^2 \Delta_h(x), \tag{7}$$

where we write $\Delta_h(x)$ for the nonnegative second-order elliptic operator in ∂X, depending parametrically on x, and given by

$$\Delta_h(x) \equiv -(\det h)^{-1/2} \frac{\partial}{\partial y_i} (\det h)^{1/2} h^{ij} \frac{\partial}{\partial y_j}. \tag{8}$$

Here, the matrix h^{ij} is the inverse of $(n-1) \times (n-1)$ matrix h appearing in (12). The operator P is self-adjoint on $L^2(X, dV_g)$. We note that this Hilbert space admits the decomposition, relative to a boundary-defining function x and product decomposition of $X \backslash U$, as

$$L^2(X, dV_g) = L^2(U, dV_g) \oplus L^2(\partial X \times [0, \epsilon],\ x^{-n} \sqrt{\det(h)}\ dx\ d^{n-1}y). \tag{9}$$

We next consider the model manifold $(X \backslash U, g_0)$. In the coordinates (x, y), the (formal) model operator P_0 has the form

$$P_0 = -x^n \frac{\partial}{\partial x} x^{2-n} \frac{\partial}{\partial x} + x^2 \Delta_{h_0}, \tag{10}$$

where we denote by $\Delta_{h_0} = \Delta_h(0)$ the nonnegative Laplace-Beltrami operator on ∂X with the induced metric $h_0(y, dy)$ as in (5). In order to obtain a self-adjoint operator P_0 on the Hilbert space $L^2(\partial X \times [0, \epsilon], dV_{g_0})$, where dV_{g_0} is the volume form for the metric g_0 in (13), we impose Dirichlet boundary conditions on the surface $x = \epsilon$.

It will be convenient to map $X \backslash U = \partial X \times [0, \epsilon]$ onto $\partial X \times [-\log\ \epsilon, \infty)$ through a change to geodesic coordinates by the transformation $r \equiv -\log x$. Near $X \backslash U$, the metric g becomes

$$g = \ dr^2 + e^{2r} h(e^{-r}, y, dy) \text{ and } dV_g = e^{r(n-1)} \sqrt{\det(h)(x,y)}\ dr d^{n-1} y. \tag{11}$$

In these coordinates, the model metric g_0 becomes

$$g_0 = \ dr^2 + e^{2r} h_0(y, dy) \text{ and } dV_{g_0} = e^{r(n-1)} \sqrt{\det(h_0)(y)}\ dr d^{n-1} y. \tag{12}$$

Let us rescale the r variable so that $\epsilon = e^{-1}$. This is effected by a change of the boundary-defining function to $u = (\epsilon e)^{-1} x$. One can easily check that the metrics g and g_0 have the same forms as in (4) and (5) with a new function $\tilde{h}$. The metric induced on ∂X by taking $\tilde{h}$ at $u = 0$ is conformally equivalent to the metric h_0.

It is convenient for calculations involving only the model operator P_0 to make one further unitary change of Hilbert space to remove the factor $e^{(n-1)r}$ from the volume forms appearing in (11) and (12). We change to the Hilbert space $L^2(\partial X \times [1, \infty), dr\ dV_{h_0}(y))$. The volume form $dV_{h_0} = \sqrt{\det(h_0)}\ d^{n-1} y$ is the one on the compact manifold ∂X associated with the metric h_0. After performing the standard unitary transformations, the model Laplacian P_0 on $L^2(\partial X \times [1, \infty), dr\ dV_{h_0})$ has the form

$$P_0 = -\partial_r^2 + e^{-2r} \Delta_{h_0} + c_n^2, \tag{13}$$

where $c_n = (n-1)/2$, and Δ_{h_0} is the operator

$$\Delta_{h_0} = -(\det\ h_0)^{-1/2} \frac{\partial}{\partial y_i} (\det\ h_0)^{1/2} h_0^{ij} \frac{\partial}{\partial y_j}, \tag{14}$$

and Dirichlet boundary conditions are imposed at $r = 1$. This form is convenient for calculations because the operator P_0 is separable.

For comparison with the operator P, however, we must make a unitary change of Hilbert space so that the Hilbert space associated with the end $X \backslash U$ can be naturally identified with a subspace of $L^2(X, dV_g)$ as in (9). For this, it suffices to define the unitary map V as $V : L^2(X \backslash U, dr dV_{h_0}) \to L^2(X \backslash U, dr dV_h)$, by $V f = \{ \det\ h_0 / \det\ h \}^{1/4} f$. We define the operator $\tilde{P}_0$, unitarily equivalent to P_0, by $\tilde{P}_0 = V P_0 V^{-1}$. We can now compare the operators P and $\tilde{P}_0$ acting on functions in $L^2(X \backslash U, dV_g) = L^2(\partial X \times [1, \infty), \sqrt{\det(h)}\ dr\ d^{n-1} y)$. Let χ be a smooth function of compact support in X so that $(1 - \chi)$ is supported in the end $X \backslash U$. It follows from Hypothesis 1 and a simple calculation that the coefficients of $(P - \tilde{P}_0)(1 - \chi)$ are $\mathcal{O}(e^{-(n-1+\epsilon)r})$. We will denote the difference $P - \tilde{P}_0$ by R_1. The estimates mentioned above imply that for any $f \in S(I\!R)$, the operators $R_1 (1 - \chi) f(P)$, and $R_1 f(P_0)(1 - \chi)$, are trace class on $L^2(X, dV_g)$.

3.3. The S-Matrix

The S-matrix $S(k)$, for $k \in I\!R$, is defined through the asymptotics of the solution of a Dirichlet problem at infinity. Let $f \in C_0^\infty(\partial X)$. Except for possibly a countable set of exceptional points k, there exists a unique solution u of the problem

$$(\Delta - k^2 - \left(\frac{n-1}{2}\right)^2)u = 0, \quad \text{in } X, \tag{15}$$

having the asymptotic form

$$u(x,y) \sim x^{\frac{n-1}{2}-ik} f_+(x,y) + x^{\frac{n-1}{2}+ik} f_-(x,y), \tag{16}$$

as $x \to 0$, and $f_-(0,y) = f$. The S-matrix is defined as the map

$$S(k)f = f_+(0,y). \tag{17}$$

In a recent work of Joshi and Sà Barreto [13], the S-matrix is computed in terms of a boundary-defining function x and metric as in (4). Let Δ_{h_0} be the nonnegative Laplace-Beltrami operator on ∂X with the metric h_0. They prove that if $|h(x, ydy) - h_0(y, dy)| = x^r f(y)$, then

$$S(k) = 2^{2ik} \frac{\Gamma(ik)}{\Gamma(-ik)} \Delta_{h_0}^{-ik} \bmod OPS^{2\Im k - r}(\partial X). \tag{18}$$

As for the model manifold with the product metric (5), it is easy to compute (cf. [8, 13]) the S-matrix exactly. It is given by

$$S_0(k) = 2^{2ik} \frac{\Gamma(ik)}{\Gamma(-ik)} \Delta_{h_0}^{-ik} + R_\infty(k). \tag{19}$$

The remainder operator $R_\infty \in OPS^{-\infty}(\partial X)$ is exactly computable in terms of a ratio of modified Bessel functions. The model Hilbert space $L^2(\partial X \times [1,\infty), dr\sqrt{\det(h_0)}\, d^{n-1}y)$ admits a direct sum decomposition relative to the spectral resolution of the Laplace-Beltrami operator Δ_{h_0}. Let $\{\lambda_j^2 \geq 0\}$ denote the eigenvalues of Δ_{h_0}. Let $c_0(k)$ denote the common factor $c_0(k) = 2^{2ik}\Gamma(ik)\Gamma(-ik)^{-1}$. The operator $R_\infty(k)$, restricted to the eigenspace of Δ_{h_0} corresponding to eigenvalue λ_j^2, is the operator of multiplication by

$$R_\infty(k)_j = -c_0(k)\lambda_j^{-2ik} \left(\frac{2\sin(ik\pi)}{\pi}\right) \left(\frac{K_{ik}(\lambda_j)}{I_{-ik}(\lambda_j)}\right). \tag{20}$$

Notice that $S_0(k)$ can also be written as

$$S_0(k) = c_0(k)\Delta_{h_0}^{-ik} \left(\frac{I_{ik}(\sqrt{\Delta_{h_0}})}{I_{-ik}(\sqrt{\Delta_{h_0}})}\right). \tag{21}$$

¿From this expression, it follows that

$$S_0(k)^{-1} = c_0(k)^{-1}\Delta_{h_0}^{ik} + R_\infty(-k). \tag{22}$$

It will be convenient to introduce another operator $\tilde{R}_\infty(k)$ defined by

$$\begin{aligned} \tilde{R}_\infty(k) &= c_0(k)^{-1}\Delta_{h_0}^{ik} R_\infty(k) \\ &= \frac{2\sin(ik\pi)}{\pi} \left(\frac{K_{ik}(\sqrt{\Delta_{h_0}})}{I_{ik}(\sqrt{\Delta_{h_0}})}\right). \end{aligned} \tag{23}$$

In terms of this operator, the inverse of $S_0(k)$ can be written as

$$S_0(k)^{-1} = (1 + \tilde{R}_\infty(k))c_0(k)^{-1}\Delta_{h_0}^{ik}. \tag{24}$$

Let $R_n(k) \in OPS^{2\Im k-(n-1+\epsilon)}(\partial X)$ denote the remainder in the formula (18) for $S(k)$. It follows from (23)–(24) that $S_0(k)^{-1}$ is a paramatrix for $S(k)$ and that

$$\begin{aligned} S_0(k)^{-1}S(k) - 1 &= c_0(k)^{-1}\Delta_{h_0}^{ik}R_n(k) + \tilde{R}_\infty(k)c_0(k)^{-1}\Delta_{h_0}^{ik}R_n(k) + \tilde{R}_\infty(k) \\ &= T_1(k) + T_2(k) + T_3(k). \end{aligned} \tag{25}$$

This formula, initially valid for $\Im k = 0$, can be extended to $\Im k \geq 0$. The operator $T_3(k)$ is a meromorphic, trace-class valued operator for $\Im k \geq 0$, with at most a finite number of poles in $\Im k > 0$, and none on the real k-axis. Following the work of [6], one can compute an explicit formula for $R_n(k)$ in terms of certain limits of generalized eigenfunctions for P. ¿From this formula, one proves that $T_1(k)$ and $T_2(k)$ are trace-class, operator-valued functions on $\Im k > 0$, and continuous on $\Im k = 0$. We discuss these terms further in section 6. Finally, the product $S_0(k)^{-1}S(k)$ is unitary for $\Im k = 0$. Consequently, with reference to the discussion at the end of section 2, we study the relative scattering phase defined through $\log(\det(1 + T_1(k) + T_2(k) + T_3(k)))$, for $\Im k = 0$. As for the behavior in $\Im k > 0$, we compute the growth of the determinant through a singular-value estimate. These estimates will yield the upper bound for $\nu(k)$, as shown in section 2.

4. Bounds on the Scattering Phase

We prove an upper bound on the scattering phase for a geometrically defined pair of operators (P_0, P) introduced in section 3. These operators on the manifold X are associated with the metric g having the form (4), and the model metric g_0 as in (5), respectively. We define an operator I as the restriction operator $I : L^2(X, dV_g) \rightarrow L^2(X\backslash U, dV_g)$, and I^* is the corresponding extension operator given by the identity map. Given Hypothesis 1, it is well-known in that we have a trace formula,

$$Tr(f(P) - I^* f(P_0)I) = -\int f'(\lambda)\xi(\lambda)d\lambda. \tag{26}$$

The function $\xi(\lambda)$ is the Krein spectral shift function.

Theorem 4.1. *Under Hypothesis 1 on the pair (g_0, g), the Krein spectral shift function $\xi(\lambda)$ is the relative scattering phase $s(k)$ for the pair (P_0, P), and satisfies the estimate,*

$$|\xi(\lambda)| \leq C_n(|\lambda|^n + 1), \tag{27}$$

for some finite constant $C_n > 0$, independent of λ, and depending on the dimension n.

The identification of the Krein spectral shift function for the pair of Laplace-Beltrami operators (P_0, P), described in section 3, with the relative scattering phase for this pair, follows from an application of the Maass-Selberg relation. Here, we sketch the idea behind the proof of the upper bound on the function $\xi(\lambda)$.

In order to obtain the upper bound in Theorem 4.1, we modify the method of Robert [23] (see also Christiansen [2]) in order to apply it to operators P of the type encountered here. There are two difficulties in treating the pair (P_0, P). First, as discussed in section 3, for any smooth function χ with $U \subset \text{supp } \chi$, we have $(P - P_0)(1-\chi) = R_1(1-\chi)$, and the coefficients of the second-order operator R_1 decay according to Hypothesis 1. Secondly, we do not have an exact commutator relation like $[P_0, A_0] = P_0$, as in [23]. Rather, we construct a skew-adjoint conjugate operator A_0 satisfying

$$[P_0, A_0] = P_0 + R_N, \tag{28}$$

where $R_N \in OPS^{-N}$, for arbitarily large N. We will use P_0 to denote either the operator P_0, as in (19), acting on $L^2(X\backslash U, x^{-n}dx\ dV_{h_0})$, with Dirichlet boundary conditions on ∂U, or the operator $\tilde{P}_0$, unitarily equivalent to P_0, acting on $L^2(X\backslash U, x^{-n}dx\ dV_h)$, with Dirichlet boundary conditions on ∂U.

Our goal is to estimate $Tr\{E_P(I_\lambda) - I^* E_{P_0}(I_\lambda) I\}$, for large λ, where $E_P(I_\lambda) = \chi_\lambda(P)$, and χ_λ is the characteristic function of an interval $I_\lambda = [\lambda_0, \lambda]$, for some fixed $\lambda_0 > 0$. By standard arguments [23], we can pass from a smooth functon f of compact support, and the operators $f(P)$ and $f(P_0)$, to the spectral projectors for P, and P_0, respectively, and the interval I_λ. Let V be a precompact set with $U \subset V$. Let χ_U be a smooth cut-off function with compact support contained in V and $\chi_U \mid U = 1$. For a function $f \in S(I\!R^+)$ supported away from 0, we write

$$\begin{aligned} Tr\{f(P) - I^* f(P_0) I\} &= Tr\{\chi_U(f(P) - I^* f(P_0) I)\} \\ &\quad + Tr\{(1-\chi_U)(f(P) - I^* f(P_0) I)\}. \end{aligned} \tag{29}$$

We will deal with each piece of the trace separately. We call the term containing χ_U the interior trace term, and the term with $(1-\chi_U)$ the exterior trace term.

It is a standard result that the compactness of the support of χ_U, and the ellipticity of P and P_0 away from ∂X, insure that the operators $\chi_U f(P)$ and $I^* \chi_U f(P_0) I$ are both in the trace class. Well-known asymptotics for the counting function for elliptic operators on compact manifolds allows us to conclude that

$$|Tr\{\chi_U E_P(I_\lambda) - I^* \chi_U E_{P_0}(I_\lambda) I\}| \ \leq\ C|\lambda|^n. \tag{30}$$

The exterior trace term is written as

$$\begin{aligned} &Tr\{(1-\chi_U)(f(P) - I^* f(P_0) I)\} \\ &\quad = Tr\{(1-\chi_U)(PP^{-1} f(P) - I^* P_0 P_0^{-1} f(P_0) I\} \\ &\quad = Tr\{P(1-\chi_U) g(P) - I^* P_0 (1-\chi_U) g(P_0) I\} \\ &\qquad + Tr\{[P, \chi_U] g(P) - I^* [P_0, \chi_U] g(P_0) I\}, \end{aligned} \tag{31}$$

where $g(x) = x^{-1} f(x)$. It follows from the fact that the commutators $[\chi_U, P]$ and $[\chi_U, P_0]$ have compact support, that each term of the second factor on the right in (31) is trace class. By an argument similar to the one used for the interior term, each term is bounded above by $|\lambda|^n$. We now concentrate on the leading term,

$$Tr\{P(1-\chi_U) g(P) - I^* P_0 (1-\chi_U) g(P_0) I\}. \tag{32}$$

Following Robert [23], this term can be written as

$$\begin{aligned} & Tr\{P(1-\chi_U)g(P) - I^* P_0(1-\chi_U)g(P_0)I\} \\ = \; & Tr\{A_0[\chi_U, P]P^{-1}f(P) - A_0[\chi_U, P_0]P_0^{-1}f(P_0)\} + R_4, \end{aligned} \quad (33)$$

where the error term R_4 is given by

$$\begin{aligned} R_4 \; = \; & Tr\{R_1(1-\chi_U)P^{-1}f(P))\} \\ & + Tr\{R_1[A_0, \chi_U]P^{-1}f(P)\} \\ & + Tr\{[A_0, R_1(1-\chi_U)]P^{-1}f(P)\} \\ & + Tr\{R_N(1-\chi_U)(P_0^{-1}f(P_0)I - P^{-1}f(P))\}. \end{aligned} \quad (34)$$

The operator A_0 is the conjugate operator for P_0 constructed in the next section.

In order to derive this representation, we use the fact that $R_1(1-\chi_U) = (P - P_0)(1-\chi_U)$, so we can write a localized commutator relation for P similar to (28). This relation has the form,

$$[P_0, A_0](1-\chi_U) = [P, A_0](1-\chi_U) + (P - P_0)[A_0, \chi_U] + [A_0, R_1(1-\chi_U)]. \quad (35)$$

Simple manipulations lead to the expression

$$\begin{aligned} & Tr\{P(1-\chi_U)g(P) - I^* P_0(1-\chi_U)g(P_0)I\} \\ = \; & Tr\{P_0(1-\chi_U)g(P) - I^* P_0(1-\chi_U)g(P_0)I\} + Tr\{R_1(1-\chi_U)g(P)\} \\ = \; & Tr\{[P_0, A_0](1-\chi_U)g(P) - I^*[P_0, A_0](1-\chi_U)g(P_0)I\} + R_2 \\ = \; & Tr\{[P, A_0](1-\chi_U)g(P) - I^*[P_0, A_0](1-\chi_U)g(P_0)I\} + R_4, \end{aligned} \quad (36)$$

where the remainder term R_4 is given in (36). As for the first term in (53), we use the trick introduced by Robert [23] in order to rewrite this term as the trace of an operator localized to a compact region,

$$\begin{aligned} & Tr\{[P, A_0](1-\chi_U)g(P) - I^*[P_0, A_0](1-\chi_U)g(P_0)I\} \\ = \; & Tr\{A_0[\chi_U, P]g(P) - A_0[\chi_U, P_0]g(P_0)\}. \end{aligned} \quad (37)$$

The form of the exterior term given above follows from this.

¿From standard trace estimates as used in the estimate for the interior term, we derive the estimate, $|Tr\{A_0[P, \chi_U]g(P) - A_0[P_0, \chi_U]g(P_0)I\}| \leq C_0|\lambda|^n$. The remainder R_4 consists of four pieces. The first three terms are controlled by the rapid decay of R_1 as the boundary coordinate $x \to 0$. Finally, we consider the term $Tr\{I^* R_N(1-\chi_U)(g(P_0)I - g(P))\}$. The operator R_N has arbitrary large negative order (see the next section). Using the functional calculus, we extract the decay of $(P - P_0)$ from $(g(P) - g(P_0))$. The operator $R_N(1-\chi_U)(P - P_0)$ is trace class, and the trace is bounded above, independent of λ. The proof of Theorem 4.1 now follows from these estimates.

5. Escape Function and Commutator Estimates

We mention how to compute a conjugate operator A_0 for the operator P_0, as in (28), and prove that the remainder R_N has the correct properties. We begin by studying the classical system with Hamiltonian $h(r,\xi) = \xi^2 + e^{-2r}p + c_n^2$, where $c_n = (n-1)/2$, and $p > 0$ is a real parameter. The Hamiltonian function h generates the geodesic flow on the cotangent bundle $T^*I\!R \sim I\!R \times I\!R$.

Proposition 5.1. *Any function $a_0(r,\xi)$, with $(r,\xi) \in I\!R^2$, satisfying the relation $\{h, a_0\} = h$, has the form*

$$a_0(r,\xi) = \frac{\sqrt{h}}{4\sqrt{h - c_n^2}}\ log\ \left(\frac{\sqrt{h - c_n^2} + \xi}{\sqrt{h - c_n^2} - \xi}\right) + \Phi(h), \tag{38}$$

for an arbitrary function Φ of h.

Proof. It is easy to check that $a_0(r,\xi)$ has the desired property. Note that since $h - c_n^2 > \xi^2$, for $p \neq 0$, the function $a_0(r,\xi)$ is well-defined. To show that any function a_0 satisfying $\{h, a_0\} = h$ has this form, we study the dynamical system generated by Hamiltonian h. The equations of motion are

$$\dot{r}(t) = \{h, r\} = 2\xi(t) \tag{39}$$
$$\dot{\xi}(t) = \{h, \xi\} = 2pe^{-2r(t)}. \tag{40}$$

The Hamiltonian is a constant of the motion. Let the initial point at $t = 0$ be (r_0, ξ_0), and define h_0 to be the energy associated with the geodesic. We also define $\phi_0 \equiv \sqrt{h_0 - c_n^2}$. Then, the solutions are

$$r(t) = \log(cosh\{2t\phi_0 + C_1\}) + C_2, \tag{41}$$
$$\xi(t) = \phi_0\ tanh\{2t\phi_0 + C_1)\}, \tag{42}$$

where the constants C_1 and C_2 are determined by the initial conditions, $tanhC_1 = \xi_0\phi_0^{-1}$, and $C_2 = -\log\{\phi_0 p^{-1/2}\}$. We now determine $a_0(r,\xi)$ through the identity

$$a_0(t) - a_0(0) = \int_0^t \dot{a_0}(s)ds = \int_0^t \{h, a_0\}ds = \int_0^t h(r(s), \xi(s))ds = h_0 t, \tag{43}$$

that follows from Hamilton's equations of motion, the Poisson bracket condition, and the conservation of energy. It is important to note that the equations for the trajectory $(r(t), \xi(t))$ can be solved explicitly for t as a function of the initial point. □

Given this consrtuction, it is easy to verify that for coordinates (r, y) on $I\!R \times \partial X$, the escape function $a_0((r,y),(\xi,\eta))$ has the identical form as in (38) where now the Hamiltonian is

$$h((r,y),(\xi,\eta)) = \xi^2 + e^{-2r} \sum_{i,j=1}^{n-1} h_{ij}(y)\eta_i\eta_j + c_n^2. \tag{44}$$

We now want to construct operator A_0 so that the commutator $[P_0, A_0]$ is equal to P_0 modulo a remainder $R_N \in OPS^{-N}$. To this end, we first construct the Weyl quantization of the symbol a_0. The Weyl quantization yields the operator A_0' acting on $u \in C_0^\infty$,

$$A_0'u(r,y) = \int e^{i(r-r')\xi + i(y-y')\cdot\eta}\ a_0((\frac{r+r'}{2}, \frac{y+y'}{2}), (\xi,\eta))u(r',y')dx'dy'd\xi d\eta. \tag{45}$$

Weyl quantization is useful because the commutator has a remainder term that is one order lower than in the usual quantization (althought this is not essential). In particular, the usual Weyl calculus yields

$$[P_0, A_0'] = P_0 + R_0, \tag{46}$$

acting on functions localized near the boundary at infinity, where $R_0 \in OPS^0$.

We need to improve this error estimate through iterations. For simplicity, we write $w = (r, y)$ and $\psi = (\xi, \eta)$. The Weyl calculus yields an asymptotic expansion for the symbol of R_0,

$$\sigma(R_0)((r,y),(\xi,\eta)) \sim \sum_{2j+1, j\geq 1} \frac{1}{(2j+1)!} \{h, a_0\}_{2j+1}((r,y),(\xi,\eta)). \tag{47}$$

The repeated Poisson bracket $\{h, a_0\}_{2j+1}$ is defined by

$$\begin{aligned} &\{h, a_0\}_{2j+1}(u,v) \\ &= 2\left(\frac{-i}{2}\right)^{2j+1} (\partial_u \cdot \partial_\psi - \partial_w \cdot \partial_v)^{2j+1} h(w,\psi) a_0(u,v)|_{w=u,\psi=v}. \end{aligned} \tag{48}$$

We now continue the construction of the escape function. We look for a symbol $a_{-1} \in S^{-1}$ so that

$$\{h, a_{-1}\} = \sigma_{pr}(R_0). \tag{49}$$

Here, we denote the principal symbol of R_0 by $\sigma_{pr}(R_0)$. The explicit formulas for the geodesic flow insure that we can solve this first-order, linear, nonhomegenous equation by integrating along the characteristics. Let A_{-1} be the Weyl quantization of the symbol a_{-1}. It then follows that

$$[P_0, A_0' - A_{-1}] = P_0 + R_{-2}, \tag{50}$$

where $R_{-2} \in OPS^{-2}$. We now repeat this iteration j-times, where $j > n/2 + 1$.

6. Bounds on Singular Values

The final major component in the proof of the upper bound on the resonance counting function is the proof of bounds on the function $F(k) = \det(1 + T(k))$ in the upper-half plane. Let us recall that the form of the inverse of the absolute S-matrix for the model problem $S_0(k)$ can be expressed as

$$S_0(k)^{-1} = (\tilde{R}_\infty + 1) c_0(k)^{-1} \Delta_{h_0}^{ik}, \tag{51}$$

where the smoothing operator $\tilde{R}_\infty(k)$ has an explicit spectral decomposition on $L^2(\partial X, dV_{h_0})$ in terms of modified Bessel functions as in (23). The coefficient $c_0(k)$ is defined before (20). Given this formula, and the formula (18) for the absolute S-matrix, we can write

$$T(k) = (S_0(k)^{-1} S(k) - 1) = \sum_{i=1}^{3} T_i(k), \tag{52}$$

where the operator-valued functions $T_i(k)$ have the form:

$$
\begin{aligned}
T_1(k) &= c_0(k)^{-1}\Delta_{h_0}^{ik}R_n(k), && (53)\\
T_2(k) &= \tilde{R}_\infty(k)c_0(k)^{-1}\Delta_{h_0}^{ik}R_n(k), && (54)\\
T_3(k) &= \tilde{R}_\infty(k). && (55)
\end{aligned}
$$

The singular values of $T_2(k)$ and $T_3(k)$ are the most easy to treat because of the explicit formulas given in section 3.2 for the eigenvalues of $\tilde{R}_\infty(k)$, and the discussion of $R_n(k)$ given below. Consequently, we will not discuss these terms any further.

In order to estimate the singular values of $T_1(k)$, the main term in (32), we need an explicit formula for $R_n(k)$. As in [6], one can obtain an explicit formula for $R_n(k)$ in terms of the analogue of the Eisenstein functions and the model resolvent. To see this, we write a localization formula expressing $R(k)$ in terms of $R_0(k)$. Let χ_i, for $i = 1, 2$, be two smooth functions localized near ∂X, with support in $X\backslash U$, and smooth up to the boundary. We can write

$$
\begin{aligned}
\chi_i R(k)\chi_j &= \chi_i R_0(k)\chi_j \\
&\quad -R_0(k)[\chi_i, P_0]R_0(k)[P_0, \chi_j]R_0(k) \\
&\quad +R_0(k)[\chi_i, P_0]R(k)[P_0, \chi_j]R_0(k) \\
&\quad \mathcal{R}(k). && (56)
\end{aligned}
$$

The remainder term $\mathcal{R}(k)$ depends explicitly on the difference R_1 and the resolvents. Consequently, the coefficients of this term vanish more rapidly than the others in (56), and they will not affect the asymptotics. We won't consider $\mathcal{R}(k)$ further but refer to [4]. The boundary-values of the resolvents are controlled by a limiting absorption principle proved via the Mourre theory using the conjugate operstor A_0 constructed in section 5. We can now take weighted limits of the terms in (56). We obtain a representation of the generalized eigenfunctions for P in terms of those for P_0, and proceed as in [6].

We present a result that can be applied to more general situations than those discussed here. In particular, this result can be used to prove the sectorial bounds on the resonance counting function mentioned in the introduction. The following result holds for certain integral operators in $\mathbb{R}^n$ that are obtained by localizing to a coordinate patch. It is then seen to be sufficient to prove bounds on the singular values of operators on $L^2(\mathbb{R}^n)$ of the form $C_0(k)\chi_0\Delta_{h_0}^{ik}\chi_1 K\chi_2$. Here, k is a complex number with $\Im k > 0$, and the functions χ are cutoff functions, and K is an integral operator (depending on k) whose kernel satisfies the bounds given below. The Laplacian here is the Laplacian Δ_{h_0} extended smoothly to $\mathbb{R}^n$.

Lemma 6.1 *Let $k \in \mathbb{C}$ with $\Im k > 0$ and $k \notin i(n/2 + \mathbb{Z})$. Let χ_0,χ_1 and χ_2 be cutoff functions on $\mathbb{R}^n$, $n \geq 2$, where χ_0 and χ_2 have compact support, and χ_1 is identically one in a neighbourhood of ∂X. Assume that the cutoff functions are analytic to order $2p$, where $p = \max\{n, [\Im k]\}$, ($[l]$ denotes the integer part of l) that is,*

$$|D^\alpha \chi| \leq C(Cp)^{|\alpha|}, \qquad (57)$$

for all multiindices α with $|\alpha| \leq 2p$. Let $K(y,z;k)$ be an integral kernel satisfying the bounds

$$|D_y^\alpha K(y,z;k)| \leq |C_1(k)| \langle y \rangle^{-2Imk-n-|\alpha|} \tag{58}$$

for $y \in \mathrm{supp}(\chi_1)$, $z \in \mathrm{supp}(\chi_2)$ and $|\alpha| \leq 2n+2$. Let q be an integer with $q+\Im k-p \geq 0$. The singular values of $C_0(k)\chi_0\Delta^{ik}\chi_1 K\chi_2$ satisfy the bound

$$\mu_j(C_0(k)\chi_0\Delta^{ik}\chi_1 K\chi_2) \leq C \left| \frac{\Gamma(ik-q+n/2)}{\Gamma(-ik+q)} C_0(k)C_1(k) \right| \left(\frac{C\langle k \rangle}{j^{1/n}} \right)^{2p}. \tag{59}$$

It follows from this estimate, and Weyl's convexity theorem relating eigenvalues to singular values (cf. [25]), that we can prove an exponential bound on $F(z)$ away from the real axis.

Sketch of the Proof. Let Δ_D denote the Laplacian with Dirichlet boundary conditions on some ball containing the support of χ_0. Then

$$\begin{aligned} \mu_j(\chi_0\Delta^{ik}\chi_1 K\chi_2) &= \mu_j(\Delta_D^{-p}\Delta_D^p \chi_0\Delta^{ik}\chi_1 K\chi_2) \\ &\leq \mu_j(\Delta_D^{-p}) \|\Delta^p \chi_0 \Delta^{ik}\chi_1 K\chi_2\| \\ &\leq C(Cj)^{-2p/n} \|\Delta^p\chi_0\Delta^{ik}\chi_1 K\chi_2\|. \end{aligned} \tag{60}$$

The constants depend only on the diameter of the support of χ_0. We must now bound the norm on the right of (60). We write

$$\|\Delta^p\chi_0\Delta^{ik}\chi_1 K\chi_2\| \leq \|\Delta^p\chi_0\Delta^{ik-q}\langle y\rangle^{-2\Im k-2q}\| \|\langle y\rangle^{2\Im k+2q}\Delta^q\chi_1 K\chi_2\|. \tag{61}$$

To bound the second term on the right in (61), we commute Δ^q past χ_1 giving two terms. The first one is $\langle y\rangle^{2\Im k+2q}\chi_1\Delta^q K\chi_2$. The assumptions on the integral kernel for K imply that this first operator has an integral kernel that is Hilbert Schmidt, and hence satisfies the bound, $\|\langle y\rangle^{2\Im k+2q}\chi_1\Delta^q K\chi_2\| \leq CC_1(k)$. The second term arising from the commutation is $\langle y\rangle^{2\Im k+2q}[\Delta^q,\chi_1]K\chi_2$. Since the derivatives of χ_1 have compact support, the norm of this term is bounded by $CC_1(k)$. Combining the preceding two estimates yields $\|\langle y\rangle^{2\Im k+2q}\Delta^q\chi_1 K\chi_2\| \leq CC_1(k)$. Now we estimate the first term on the right of (61). For $k \notin i(n/2+\mathbb{Z})$ the operator whose norm we must bound has integral kernel

$$\begin{aligned} &(\Delta^p\chi_0\Delta^{ik-q}\langle y\rangle^{-2\Im k-2q})(x,y) \\ &\quad = \; C2^{2ik}g(k)\Delta_x^p\chi_0(x)|x-y|^{-2ik+2q-n}\langle y\rangle^{-2\Im k-2q}, \end{aligned} \tag{62}$$

where $g(k) = \Gamma(ik-q+n/2)\,\Gamma(-ik+q)^{-1}$. This kernel has to be analysed carefully near and away from the diagonal. The off-diagonal piece is easily seen to be Hilbert Schmidt. It is in the analysis of the kernel near the diagonal the we use the partial analyticity of the cut-off functions. □

References

[1] U. Bunke, M. Olbrich, *Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group*, Ann. Math. **149**, 627–689 (1999).

[2] T. Christiansen, *Spectral asymptotics for compactly supported perturbations of the Laplacian on* $I\!R^n$, Commun. PDE **23**, 933–948 (1998).

[3] R. G. Froese, *Upper bounds for the resonance counting function for the Schrödinger operators in odd dimensions*, Canadian J. Math. **50**, 538–546 (1ç1998).

[4] R. G. Froese, P. D. Hislop, *Upper bounds for the resonance counting function for some asymptotically hyperbolic manifolds*, in preparation.

[5] R. G. Froese, P. D. Hislop, P. A. Perry, *A Mourre Estimate and Related Bounds for the Laplace Operator on a Hyperbolic Manifold with Cusps of Nonmaximal Rank*, J. Funct. Anal. **98**, 292-310 (1991).

[6] R. G. Froese, P. D. Hislop, P. A. Perry, *The Laplace operator on hyperbolic Manifolds with Cusps of Non-maximal Rank*, Inventiones Math. **106**, 295-333 (1991).

[7] C. Gérard, A. Matrinez, *Principe d'absorption limite pour les opérateurs de Schrödinger à longue portée*, C. R. Acad. Sci. Paris **306**, 121–123 (1988).

[8] P. D. Hislop, *The geometry and spectra of hyperbolic manifolds*, Proc. Indian Acad. Sci. (Math. Sci.) **104**, 715–776 (1994).

[9] L Guillopé, M. Zworski, *Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature at infinity*, Asymp. Anal. **11**, 1–22 (1995).

[10] L. Guillopé, M. Zworski, *Upper bounds on the number of resonances of non-compact Riemann surfaces*, J. Func. Anal. **129**, 364–389 (1995).

[11] L. Guillopé, M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. of Math. **145**, 597–660 (1997).

[12] A. Jensen, *High energy resolvent estimates for generalized many-body Schrödinger operators*, Publ. RIMS, Kyoto Univ. **25**, 155–167 (1989).

[13] M. S. Joshi, A. Sá Barreto, *Inverse scattering on asymptotically hyperbolic manifolds*, Acta Math. 2000.

[14] P. Lax, R. S. Phillips, *Scattering theory for automorphic functions*, Ann. Math. Studies **87**, Princeton: Princeton University Press, 1976.

[15] N. Mandouvalos, *Scattering operator and Eisenstein integral for Kleinian groups*, Math. Proc. Cambridge Philos. Soc. **108**, 203–217 (1990).

[16] R. Mazzeo, R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75**, 260–310 (1987).

[17] S. J. Patterson,*The Laplacian operator on a Riemann surface, I, II, and III*, Compositio math. **31**, 83–107 (1975); **32**, 71–112 (1976); **33**, 227–259 (1976).

[18] S. J. Patterson, P. A. Perry, *Divisor of the Selberg Zeta function, I. Even Dimensions*, to appear in Duke Math. J. 2000.

[19] P. A. Perry, *The Laplace operator on a hyperbolic manifold. II. Eisenstein series and the scattering matrix*, J. reine. angew. Math. **398**, 67–91 (1989).

[20] P. A. Perry, *The Selberg zeta function and a local trace formula for Kleinian groups*, J. reine. angew. Math. **410**, 116–152 (1990).

[21] P. A. Perry, *The Selberg Zeta function and scattering poles for Kleinian groups*, Bull. Amer. Math. Soc. (N. S.) **24**, 327–333 (1991).

[22] P. A. Perry, *Poisson formula and lower bounds on resonances for hyperbolic manifolds*, preprint 2000.

[23] D. Robert, *On the Weyl formula for obstacles*, in "Partial differential equations and mathematical physics", 264–285, Progress in Nonlinear Differential Equations and their Applications, Boston: Birhäuser, 1996.

[24] D. Robert, *Asymptotique de la phase de diffusion à haute énergie pours les perturbations de second ordre du Laplacien*, Ann. Scien. Ecole Norm. Sup. **25**, 107–124 (1992).

[25] G. Vodev, *Sharp bounds on the number of scattering poles for perturbations of the Laplacian*, Commun. Math. Phys. **146**, 205–216 (1992).

[26] M. Zworski, *Dimension of the limit set and the density of resonances for convex co-compact hyperbolic surfaces*, to appear in Inventionnes math. 1999.

[27] M. Zworski, *Counting Scattering Poles*, in *Spectral and Scattering Theory*, M. Ikawa, ed. , Marcel Decker, 1994.

Mathematics Department, University of British Columbia, Vancouver, BC Canada
rfroese@math.ubc.ca

Mathematics Department, University of Kentucky, Lexington, KY 40506-0027 USA
hislop@ms.uky.edu