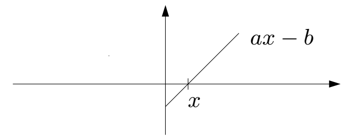
Introduction to variational methods and finite elements

1.2.3. **Variational formulations of BVP**:

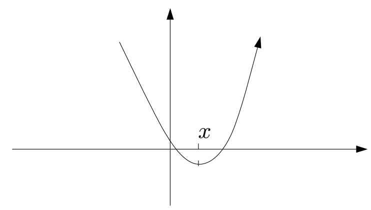


Problem: Sove $ax = b$ $\qquad x = \dfrac{-b}{a}$

Reformulate the problem:

Consider $E = \frac{1}{2}ax^2 + bx$

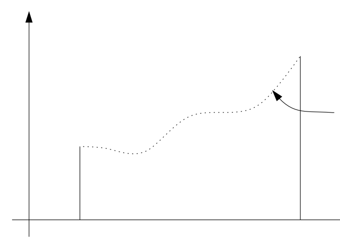
Find $x^* : E(x^*) = \min\limits_{x} E(x)$



1. **Rayleigh-Ritz Method**:

Consider a differential equation

$$\left.\begin{array}{rll} Au = & u'' = f(x) & (1a) \\ & u(0) = \alpha \quad u(1) = \beta & (1b) \end{array}\right\}$$

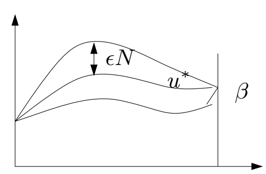


Consider the functional: $E[u] = \int\limits_0^1 \frac{1}{2}(u')^2 + fu\,dx \leftarrow$ potential energy functional.

Claim: If $u^* : E[u^*] = \min\limits_{u} E[u]$ and u^* satisfies (1b) then u^* solves (1).

Proof:

Let $u = u^* + \varepsilon\eta \leftarrow(_{\text{arbitrary}})\ \eta(0) = 0 = \eta(1)$ and $\eta \in C^2$ so we have a 1 parameter family of functions which are perturbations of u^*. The function



$$\begin{aligned} E(\varepsilon) &= E[u^* + \varepsilon\eta] = \int\limits_0^1 \frac{1}{2}\left(u^{*\prime} + \varepsilon\eta'\right)^2 + f(u^* + \varepsilon\eta)\,dx \\ E'(0) &= \int\limits_0^1 (u^{*\prime} + \varepsilon\eta')\eta' + f\eta\,dx\Big|_{\varepsilon=0} \\ &= \int\limits_0^1 (u^*)'\eta' + f\,\eta\,dx \\ &= u^{*\prime}\eta\Big|_0^1 - \int\limits_0^1 \left(u^{*\prime\prime} - f\right)\eta\,dx = 0 \end{aligned}$$

$$\Rightarrow \quad \text{Since } \eta \text{ is arbitrary we could choose } \eta = \left(u^{*''} - f\right)$$

$$\Rightarrow \quad u^{*''} - f = 0 \leftarrow \text{Euler -Lagrange equation for } E[u].$$

How is this useful? Let us assume that $U(x) = \sum_{k=1}^{N-1} \alpha_k \Psi_k(x)$

$\uparrow$ basis functions

Then

$$E(\boldsymbol{\alpha}) = \int_0^1 \frac{1}{2}\left[\sum_{k=1}^{N-1} \alpha_k \Psi_k'(x)\right]^2 + f(x)\sum_{k=1}^{N} \alpha_k \Psi_k(x)\, dx$$

For a min:

$$\begin{aligned}
0 = \frac{\partial E}{\partial \alpha_j} &= \int_0^1 \left(\sum_{k=1}^{N-1} \alpha_k \Psi_k'\right) \cdot \Psi_j' + f(x)\Psi_j(x)\, dx \\
&= \sum_{k=1}^{N-1} \alpha_k \int_0^1 \Psi_k' \Psi_j'\, dx + \int_0^1 f(x)\Psi_j(x)\, dx, \qquad j = 1, \ldots, N-1
\end{aligned}$$

$$\Rightarrow \qquad \mathbf{A}\boldsymbol{\alpha} = \mathbf{b} \text{ where } A_{kj} = \int_0^1 \Psi_k'\Psi_j'\, dx \qquad b_j = -\int_0^1 f(x)\Psi_j(x) dx.$$

Example : $\left.\begin{array}{l} u'' = -x^2 \\ u(0) = 0 = u(1) \end{array}\right\} u(x) = \frac{x}{12}(1 - x^3)$

Assume $\{\Psi_j\} = \left\{1, x, x^2, x^3\right\}$

$$\begin{aligned}
U_3(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 \\
U_3(0) &= c_0 = 0 \quad u_3(1) = c_1 + c_2 + c_3 = 0 \\
U_3(x) &= ax(1-x) + bx^2(1-x) \\
&= \underset{\Psi_1}{a\{x - x^2\}} \;\; + \underset{\Psi_2}{b\ \{x^2 \pm x^3\}}
\end{aligned}$$

$$-c_2 x + c_2 x^2 = c_3 x + c_3 x^3$$
$$-c_2 x(1-x)$$
$$c_2 = -c_1 - c_3$$

$$\begin{aligned}
A_{11} &= \int_0^1 (1-2x)^2\, dx = \int_0^1 1 - 4x + 4x^2\, dx = x - 2x^2 + \frac{4}{3}x^3\Big|_0^1 = \frac{1}{3} \\
A_{22} &= \int_0^1 (2x - 3x^2)^2 dx = \int_0^1 4x^2 - 12x^3 + 9x^4\, dx = \frac{4}{3}x^3 - 3x^4 + \frac{9}{5}x^5\Big|_0^1 = \frac{20 - 45 + 27}{15} = \frac{2}{15} \\
A_{12} &= A_{21} = \int_0^1 (1-2x)(2x - 3x^2)\, dx = \frac{1}{6} \\
b_1 &= +\int_0^1 x^2(x - x_2)\, dx = \frac{1}{20}
\end{aligned}$$

$$b_2 = \int_0^1 x^2(x^2 - x^3)dx = \frac{1}{30}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{15} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ \frac{1}{30} \end{bmatrix} \Rightarrow a = \frac{1}{15} \qquad b = \frac{1}{6}$$

$$U_3(x) = \frac{x}{30}(1-x)(2+5x)$$

Exact:

$$u_{\text{ex}}(x) = \frac{x}{12}(1-x^3)$$

Natural and Essential B-C

Notice that the basis functions $\{\Psi_i\}$ were required to satisfy the BC $u(0) = 0 = u(1)$. B-C that have to be forced onto the trial solution are called *essential B-C* – typically these involve the solution values and *not the derivatives*. In the case of derivative B-C it is possible to build the B-C into the energy functional to be minimized.

Consider $\begin{cases} u(0) = \alpha \quad u'(1) = \beta \\ u'' = f(x) \end{cases}$

Old energy functional — Boundary Condition

$$\begin{aligned} E[u] &= \int_0^1 \frac{1}{2}(u')^2 + fu\,dx && - \beta u(1) \\ \delta E[u] &= \int_0^1 u'\delta u' + f\delta u\,dx && - \beta\delta u(1) \\ &= u'\delta u\Big|_0^1 - \int_0^1 \left(u'' - f\right)\delta u\,dx && - \beta\delta u(1) \\ &= \{u'(1) - \beta\}\delta u(1) - \int_0^1 (u'' - f)\delta u\;dx = 0 \\ \Rightarrow\;& u'' = f \text{ and } u'(1) = \beta \qquad \text{but says nothing about } u(0)! \end{aligned}$$

↑ Don't have to enforce BC at this endpoint.

Eg. 2:

$$u'' = -x = f \qquad \boxed{u_{\text{exact}} = 2 + \frac{7x}{2} - \frac{x^3}{6}}$$

$$u(0) = 2 \qquad u'(1) = 3$$

↑ essential ↑ natural

Let: $U(x) = 2 + x(a_1 + a_2 x) = 2 + a_1 x + a_2 x^2$

$$\begin{aligned}
&\quad\quad\downarrow \text{satisfies homogeneous version of essential BC}\\
U_N(x) &= \alpha + \sum_{i=1}^{N} a_i\Psi_i(x) = \sum_{i=0}^{N} a_i\Psi_i(x) \qquad a_0 = 1 \quad \Psi_0(x) = 2\\
E(\mathbf{a}) &= \int_0^1 \frac{1}{2}\left(\sum a_i\Psi_i'\right)^2 + f\left(\sum a_i\Psi_i\right) dx - 3\left(\sum a_i\Psi_i(1)\right)\\
0 = \frac{\partial E}{\partial a_j} &= \int_0^1 \left(\sum a_i\Psi_i'\right)\Psi_j' + f\Psi_j dx - 3\Psi_j(1)\\
0 &= \left(\Psi_i', \Psi_j'\right) a_i + (f, \Psi_j) - 3\Psi_j(1)\\
0 &= \mathbf{A}\mathbf{a} + \mathbf{b}\\
A_{11} &= \int_0^1 dx = 1 \quad A_{21} = A_{12} = \int_0^1 2x\, dx = 1 \quad A_{22} = \int_0^1 4x^2 dx = \frac{4}{3}x^3\Big|_0^1 = \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
b_1 &= \int_0^1 (-x)\cdot x\, dx - 3 = -3\tfrac{1}{3}\\
b_2 &= \int_0^1 (-x)x^2\, dx - 3 = -3\tfrac{1}{4}
\end{aligned} \Rightarrow \begin{bmatrix} 1 & 1\\ 1 & 4/3 \end{bmatrix}\begin{bmatrix} a_1\\ a_2 \end{bmatrix} = \begin{bmatrix} 10/3\\ 13/4 \end{bmatrix} \Rightarrow a_1 = \frac{43}{12} \quad a_2 = -\frac{1}{4}$$

$$U_2(x) = 2 + \frac{43}{12}x - \frac{x^2}{4}$$

x	0	0.2	0.4	0.6	0.8	1	
RR	2	2.707	3.393	4.060	4.707	5.333	
EX	2	2.699	3.389	4.064	4.715	5.333	← Built in

2. **General calculus of variations**:

$$\begin{aligned}
I[u] &= \int_a^b F(x, y, y')\, dx \qquad\qquad y(a) = 0, \quad y(b) = 0\\
0 = \delta I &= \int_a^b \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}(\delta y')\, dx\\
&= \delta y\frac{\partial F}{\partial y}\Big|_a^b - \int_a^b \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right]\delta y\, dx\\
&\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \qquad\qquad \text{Euler-Lagrange Eq.}
\end{aligned}$$

Eg.1

$$\begin{aligned}
&- (pu')' + qu = f\\
&\quad u'(a) = \alpha \quad u(b) = \beta\\
I[u] &= \int_a^b p(u')^2 + qu^2 - 2fu\, dx + \alpha u
\end{aligned}$$

$$\begin{aligned}
&F = p(u')^2 + qu^2 - 2fu\\
&\frac{\partial F}{\partial u} = 2qu - 2f\\
&\frac{\partial F}{\partial u'} = 2pu'\\
&\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} = 2\left\{(pu')' - qu + f\right\} = 0
\end{aligned}$$

Eg. 2: **Sturm-Lioville Eigenvalue problems**:

$$u(a) = 0 = u(b);\ +(pu')' + qu + \lambda ru = 0 \tag{1}$$

Multiply by v and integrate $\int_a^b +(pu')'v + quv\,dx + \lambda\int_a^b ruv\,dx = 0$

$$\begin{aligned}
\lambda &= I[u] = \frac{\int_a^b p(u')^2 - qu^2\,dx}{\int_a^b ru^2\,dx} = \frac{I_1}{I_2} \\
\delta\lambda &= \frac{\delta I_1 I_2 - I_1 \delta I_2}{I_2^2} = \frac{1}{I_2}\left(\delta I_1 - \lambda\delta I_2\right) \\
&= \frac{\int_a^b 2pu'\delta u' - 2qu\delta u - \lambda ru\delta u\,dx}{\int_a^b ru^2\,dx} = \frac{-2\int_a^b \left[(pu')' + qu + \lambda ru\right]\delta u\,dx}{\int_a^b ru^2\,dx} = 0 \\
\therefore \delta\lambda &= 0 \quad \Rightarrow \text{ the function } u_0 \text{ which minimizes } I[u] \text{ is an eigenfunction of (1) and} \\
&\ \lambda = I[u_0] \text{ is its eigenvalue.}
\end{aligned}$$

Higher eigenvalues: $\lambda_n = \min\limits_{\substack{(u,\Psi_k)=0 \\ k=1,\dots,n-1}} I[u]$ $\longleftarrow$ constrained minimization problem.

3. **Method of weighted residuals**

What do we do for nonlinear or dissipative problems for which potential energy functionals don't exist or cannot be found easily?

Consider a BVP

$$\begin{aligned}
Lu &= f \qquad \text{in } \Omega = (a,b) && \text{(1a)} \\
u(a) &= \alpha \quad ; \quad u'(b) + \sigma u(b) = \beta && \text{(1b)}
\end{aligned}$$

An approximate solution U won't in general satisfy (1a) and we associate with U the so-called residual

$$\begin{aligned}
r(U) &= LU - f \\
\text{Note that } r(u) &= 0 \\
\text{exact} \uparrow &
\end{aligned}$$

A whole class of methods are obtained by considering various ways to minimize the residual in some sense, usually:

$$\int_a^b r(U)\phi_i\,dx = 0$$

$\uparrow$ test or weight functions

where $\quad U = \sum_{i=1}^{N} a_i \Psi_i(x) \leftarrow$ basis function

1. **Collocation**: weight functions $= \delta(x - x_i)$ basis functions polynomials:

Eg:

$$\begin{aligned} &u'' - u' = 0 \\ &u(0) = 1, \quad u(1) = 0 \end{aligned}$$

(A) Let

$$\begin{aligned} \Psi_i(x) &= \left(x^{i+1} - x\right) \text{ which satisfy the homogeneous B-C for the problem} \\ U(x) &= 1 - x + a_1\left(x^2 - x\right) + a_2\left(x^3 - x\right) \\ r(U) &= \{a_1 2 + a_2 6x\} - \left\{a_1(2x-1) + a_2\left(3x^2 - 1\right)\right\} + 1 \\ &= (3 - 2x)\, a_1 - a_2\left(3x^2 - 6x - 1\right) + 1 \end{aligned}$$

With $x_1 = 0$ and $x_2 = 1$ as collocation points we have

$$\begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow U_1(x) = 1 - x - \left(\frac{3}{11}\right)(x^2 - x) - \frac{2}{11}(x^3 - x)$$

If we choose x_1 and x_2 are chosen to be the zeros of the 2nd degree Legendre polynomial

$$U_2(x) = 1 - x - \frac{1}{4}(x^2 - x) - \left(\frac{1}{6}\right)(x^3 - x)$$

The exact solution is $u(x) = \dfrac{(e - e^x)}{(e - 1)}$.

x	$u(x)$	$U_1(x)$	$U_1 - u$	U_2	$U_2 - u$	U_3	$U_3 - u$
1/4	0.834704	0.837750	3.05×10^{-3}	0.835938	1.23×10^{-3}	0.769965	-6.47×10^{-2}
1/2	0.622459	0.636364	1.39×10^{-2}	0.625000	2.54×10^{-3}	0.506144	-1.16×10^{-1}
3/4	0.349932	0.360795		0.351563	1.63×10^{-3}	0.269965	-8.00×10^{-2}

(B) **With different trial functions**:

$$\begin{aligned} U(x) &= 1 - x + a_1 \sin \pi x + a_2 \sin 3\pi x \\ r(U) &= -\pi(\pi \sin \pi x + \cos \pi x) a_1 - 3\pi(3\pi \sin 3\pi x + \cos 3\pi x) c_2 \\ U_3(x) &= 1 - x + 0.017189 \sin \pi x + 0.011045 \sin 3\pi x \end{aligned}$$

using zeros of 2nd degree Legendre polys.

Where to collocate?

What basis fucntions to use?

DeBoor, C. and Swartz, B. , Collocation at Gaussian points, *SIAM J. Num. Anal.* **10** (1973), 582–606.

2. **Method of moments**: weight functions $\phi_i = x^i$.

Eg:

$$u'' + u + x = 0$$
$$u(0) = 0 = u(1)$$

Let
$$U(x) = a_1 x(1-x) + a^2 x^2(1-x) + \dots$$
$$r(U) = x + a_1(-2 + x - x^2) + a_2(2 - 6x + x^2 - x^3)$$
$$\int_0^1 r(U)\cdot 1\,dx = 0 \qquad \text{and} \qquad \int_0^1 r(U)\cdot x\,dx = 0$$
$$\Rightarrow \begin{bmatrix} \frac{11}{6} & \frac{11}{12} \\ \frac{11}{22} & \frac{19}{20} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$
$$U(x) = x(1-x)\left(\frac{122}{649} + \frac{110}{649}x\right)$$
$$u_{\text{ex}}(x) = \frac{\sin x}{\sin 1} - x$$

3. **Galerkin method**: 'Expand $U(x)$ and $w(x)$ in terms of the same basis functions.'

$$u(x) = \sum_{i=1}^{N} a_i \Psi_i \qquad w(x) = \sum_{i=1}^{N} b_i \Psi_i$$

$$\int_\Omega r\left(\sum_{i=1}^{N} a_i\Psi_i\right)\left(\sum_{i=1}^{N} b_i\Psi_i\right) dx = 0 \Rightarrow \int_\Omega r\left(\sum_{i=1}^{N} a_i\Psi_i\right)\Psi_j\, dx = 0$$

Since the b_i are arbitrary.

Eg.
$$u'' + u + x = 0$$
$$u(0) = 0 = u(1)$$

Let
$$U(x) = a_1 x(1-x) + a_2 x^2(1-x)$$
$$r(U) = x + a_1(-2 + x - x^2) + a_2(2 - 6x + x^2 - x^3)$$
$$\int_0^1 r(u)\cdot\Psi_1(x)dx = \int_0^1 \left\{x + a_1\left(-2 + x - x^2\right) + a_2\left(2 - 6x + x^2 - x^3\right)\right\}\cdot \begin{matrix} x(1-x) \\ x^2(1-x) \end{matrix}\; dx = 0$$
$$\Rightarrow \begin{bmatrix} \frac{3}{10} & \frac{3}{20} \\ \frac{3}{20} & \frac{13}{105} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{12} \\ \frac{1}{20} \end{bmatrix} \Rightarrow a_1 = \frac{7}{369} \quad a_2 = \frac{7}{41}$$
$$U(x) = x(1-x)\left(\frac{71}{369} + \frac{7}{41}x\right)$$
$$u_e(x) = \frac{\sin x}{\sin 1} - x$$

Relationship between Rayleigh Ritz and Galerkin:

Rayleigh Ritz $\qquad u'' + u = f = -x \quad u(0) = 0 = u(1)$

$$
\begin{aligned}
I[u] &= \int_0^1 u'^2 - u^2 + 2fu\,dx \\
0 = \delta I &= \int_0^1 2u'\delta u' - 2u\delta u + 2f\delta u\,dx \\
&= \left[u'\delta u\right]_0^1 - \int_0^1 \left(u'' + u - f\right)\delta u\,dx \\
\Rightarrow 0 &= \int_0^1 (u'' + u - f)\delta u\,dx \\
&= \int_0^1 r(u)\delta u\,dx
\end{aligned}
$$

$$
\begin{aligned}
I(\mathbf{a}) &= \int_0^1 \left(\sum a_i\Psi_i'\right)\left(\sum a_i\Psi_j'\right) - \left(\sum a_i\Psi_i\right)\left(\sum a_j\Psi_j\right. \\
&\quad +2f\sum a_j\Psi_j\,dx \\
0 &= \frac{\partial I}{\partial a_k} = 2\left\{\sum a_i(\Psi_i', \Psi_k') - \sum a_i(\Psi_i, \Psi_k)\right. \\
&\quad +(f, \Psi_k)\} \\
&\quad f = -x
\end{aligned}
$$

Rayleigh-Ritz and Galerkin methods are identical for this problem. − true in general for linear problems but not for nonlinear problems.

From WR to the weak form:

$$
\mathbf{WR} \Rightarrow \quad \int_0^1 (u'' + u + x)v\,dx = 0
$$

$$
u(0) = 0 = u(1) \qquad v \text{ satisfy homogeneous Dirichlet i.e. } v(0) = 0 = v(1)
$$

Say we wanted to express u in terms of functions that are not twice differentiable then we integrate by parts to throw as many derivatives as needed from u to v.

$$
\int_0^1 (u'' + u + x)v\,dx = u'v\Big|_0^1 + \int_0^1 -u'v' + uv + xv\,dx = 0.
$$

Now if we let $u = \sum a_i\Psi_i$ and $v = \sum b_i\Psi_i$ we have since v is arbitrary

$$
-\sum_i a_i(\Psi_i', \Psi_j') + \sum a_i(\Psi_i, \Psi_j) + (x, \Psi_j) = 0
$$

$$
\mathbf{Aa} = \mathbf{b}
$$

where

$$
[\mathbf{A}]_{jj} = (\Psi_i', \Psi_j') - (\Psi_i, \Psi_j); \quad \mathbf{b} = (x, \Psi_j)
$$

Identical to Rayleigh-Ritz eqs.

What about a natural BC?

$$\text{(S)} \qquad \begin{cases} u'' + u + x = 0 \\ u(0) = \alpha \quad u'(1) = \beta. \end{cases}$$

$$\int_0^1 (u'' + u + x) v \, dx = 0 \qquad \forall v \in H_0^1$$

$$u'v\Big|_0^1 - \int_0^1 u'v' - uv - xv \, dx = 0$$

$$\beta v(1) - \int_0^1 u'v' - uv - xv \, dx = 0$$

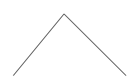
$$\text{(W)} \qquad \boxed{\begin{array}{l} \text{Find } u \in H_\alpha^1 \text{ such that} \\ \displaystyle\int_0^1 u'v' - uv - xv \, dx - \beta v(1) = 0 \quad \forall v \in H_0^1 \end{array}}$$

The Finite Element method

F.E. Model Problem

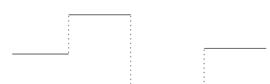
$$\left.\begin{array}{l} u'' + f = 0 \\ u(a) = G; \quad u'(b) = H \end{array}\right\} (S)$$

W Residual $\displaystyle\int_a^b \big(u'' + f\big) \, v \, dx = 0 \longleftarrow$ can't plug in

$N_i(x) =$ 

$u_h(x) = \sum\limits_{i=0}^{N} u_i N_i(x)$

$N_i''(x) = \delta(x - x_{i-1}) - 2\delta(x - x_i) + \delta(x - x_{i+1})$

$N_i'(x) =$ 

We must therefore relax the continuity required of the trial solution by going to a weak formulation of the problem.

$$u'v\Big|_a^b + \int_a^b -u'v' + fv \, dx = 0$$

$$\therefore \quad \int_a^b u'v' \, dx = Hv(b) + \int_a^b fv \, dx$$

$$a(u, v) = Hv(b) + (f, v) \qquad \longleftarrow (*)$$

(W) Given $f \in L_2$ and constants G, H find $u \in \{u \in H^1, u(a) = g\} = H_g^1$ such that *for all* $v \in H_0^1 = \{v \in H^1, v(a) = 0\}$ we have that $\int_a^b u'v' \, dx = Hv(b) + \int_a^b fv \, dx$

$\Bigg\}$ $H^k = \{u : u^{(k)} \in L^2\} \leftarrow$ Sobolov space

Claim: If u is a solution of (S) then $u$satisfies (W). Conversely, provided u is sufficiently differentiable then if u satisfies (W) it follows that u also satisfies (S).

$$(S) \Rightarrow (W)(\text{ See *})$$

$$(W) \Rightarrow (S)$$

$$\int_a^b u'v'\,dx = Hv(b) + \int_a^b fv\,dx \text{ for all } v \in H_0^1$$

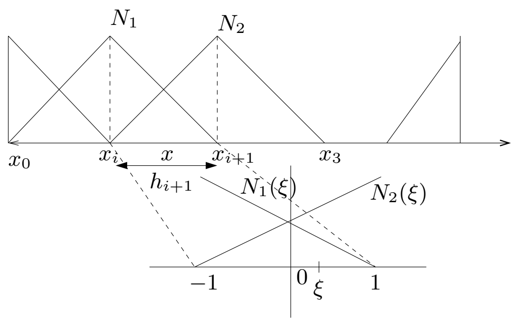
$$u'v\Big|_a^b - \int_a^b u''v\,dx = hv(b) + \int_a^b fv\,dx$$

$$\int_a^b (u''+f)v\,dx + \{H - u'(b)\}v(b) = 0$$

$$v \text{ arbitrary } \Rightarrow u'' + f = 0 \qquad u'(b) = H$$

$$\uparrow \text{natural BC}$$

Using Finite Element Basis Functions: It is useful in the calculation of the integrals to transform each of the subintervals in turn to the same standard interval – on which we construct cardinal basis functions



$$\left.\begin{array}{l} N_1(\xi) = \frac{1}{2}(1-\xi) \\ N_2(\xi) = \frac{1}{2}(1+\xi) \end{array}\right\} N_a(\xi) = \frac{1}{2}(1+\xi\,\xi_a)$$

$$\xi_a = \begin{cases} -1 & a = 1 \\ 1 & a = 2 \end{cases}$$

Transformation:

$$\xi(x) = c_1 + c_2 x \qquad \xi(x_i) = c_1 + c_2 x_i = -1 \qquad c_2 = \frac{2}{x_{i+1} - x_i} = \frac{2}{h_{i+1}}$$

$$\xi(x_{i+1}) = c_1 + c_2 x_{i+1} = 1 \quad c_1 = -\frac{(x_i + x_{i+1})}{h_{i+1}}$$

$$\xi(x) = \frac{2x - (x_i + x_{i+1})}{h_{i+1}}$$

$$\text{Similarly} \qquad x(\xi) = \frac{h_{i+1}\xi + (x_i + x_{i+1})}{2}$$

Notice we can express the transformation in terms of $N_a(\xi)$:

$$\begin{aligned}
x(\xi) &= \sum_{a=1}^{2} N_a(\xi)x_a \\
\frac{dx}{d\xi} &= \sum_{a=1}^{2} \frac{dN_a}{d\xi}(\xi)x_a = \left(\frac{dN_a}{d\xi}(\xi) = \frac{\xi_a}{2}\right) \\
&= \frac{1}{2}\sum_{a=1}^{2} \xi_a x_a = \frac{1}{2}\left(-x_i + x_{i+1}\right) = \frac{h_{i+1}}{2} \\
\frac{d\xi}{dx} &= \frac{2}{h_{i+1}}
\end{aligned}$$

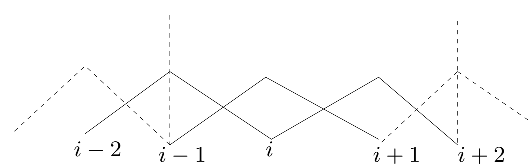
Galerkin approximation:

$$\begin{aligned}
\text{Let} \qquad u^h(x) &= GN_0(x) + \sum_{i=1}^{N} N_i(x)u_i \\
v^h(x) &= \sum_{i=1}^{N} N_i(x)v_i \longleftarrow \text{ arbitrary} \\
a\left(u^h, v^h\right) &= hv^h(b) + \left(f, v^h\right) \\
&\Rightarrow \int_a^b \left(\sum_{i=1}^{N} u_i N_i'(x)\right)\left(\sum_{j=1}^{N} v_i N_j'(x)\right)\, dx = h\sum v_j N_j(b) + \int_a^b f \sum_{j=1}^{N} v_j N_j(x)\, dx \\
v_j \text{ arbitrary} &\Rightarrow \sum_{i=1}^{N} u_i \left(N_i', N_j'\right) = hN_j(x_N) + (N_j, f) \qquad j = 1, \ldots, N
\end{aligned}$$

$$\mathbf{Ku} = \mathbf{f}$$

stiffness matrix ↑

$$\begin{aligned}
K_{ij} &= \left(N_i', N_j'\right) = \int_a^b N_i' N_j'\, dx \\
f_j &= hN_j(x_N) + (N_j, f)
\end{aligned}$$



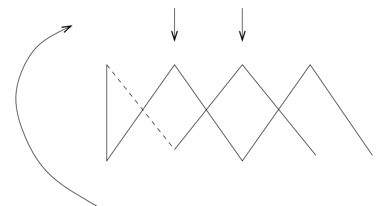
Notice :

(a) $K_{ij} = K_{ji}$ Symmetric

(b) K_{ij} is positive definite

$$\sum_{1 \le ij \le N} u_i K_{ij} u_j = \sum_{ij} u_i a(N_i, N_j) u_j = a\left(u^h, u^h\right) = \int_a^b \left(u'\right)^2\, dx \ge 0 \quad \text{pos. semi def.}$$

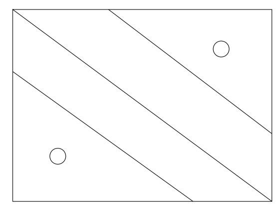
$$\left(u^h\right)' = 0$$
$$\Longrightarrow u^h = \text{const but since } u^h(0) = 0$$



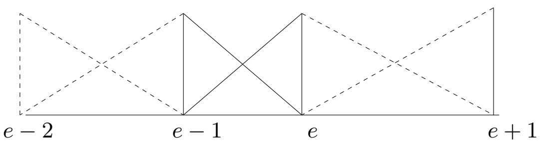
$$\Longrightarrow u^h(x) = \sum u_i N_i(x) = 0$$
$$\uparrow \text{ form a basis } \Rightarrow u_i \equiv 0.$$
$$\therefore \mathbf{u}^T\mathbf{K}\mathbf{u} \geq 0 \quad = 0 \Rightarrow \mathbf{u} = 0$$

Tridiagonal system - like finite differences.

(c) $K_{ij} = 0 \quad \text{if } |i-j| \geq 2$



The trick to doing the integrals is to calculate them element by element: *so called assembly of the stiffness matrix*:



Element stiffness matrix

$$
\begin{aligned}
K_{ij} &= \int_a^b N_i' N_j' \, dx = \sum_{e=1}^{N} \int_{x_{e-1}}^{x_e} N_i'(x) N_j'(x) \, dx \\
&= \sum_{e=1}^{N} k_{ab}^e \\
k_{ab}^e &= \int_{x_{e-1}}^{x_e} N_a'(x) N_b'(x) \, dx \\
&= \int_{-1}^{1} \frac{dN_a}{d\xi}(\xi)\xi'(x) \frac{dN_b}{d\xi}\xi'(x) \cdot \frac{dx}{d\xi} \, d\xi \\
&= \frac{2}{h_e} \frac{\xi_a}{2} \frac{\xi_b}{2} \cdot \int_{-1}^{1} d\xi = \frac{\xi_a \xi_b}{h^e} = \frac{(-1)^{a+b}}{h^e} \\
\therefore [k_{ab}^e] &= \frac{1}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } k_{ab}^e = 0 \text{ if } a, b \notin \{e-1, e\}
\end{aligned}
$$

$$
\begin{aligned}
N_a(\xi) &= \frac{1}{2}(1 + \xi \, \xi_a) \\
N_a'(\xi) &= \frac{\xi_a}{2} \\
\frac{d\xi}{dx} &= \frac{2}{x_e - x_{e-1}} = \frac{2}{h_e}
\end{aligned}
$$

Assembly:

FIGURE assume $h_e = h \quad \forall \ e$ i.e. uniform mesh

$$
\frac{1}{h} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \mathbf{F} \end{bmatrix}
$$

To calculate the force for a general $f(x)$:

$$
f_j = \int_a^b f(x) N_j(x) \, dx
$$

We assume $f(x) \simeq f^h(x) = \sum_j f_j N_j(x)$

The force vectors are also assembled

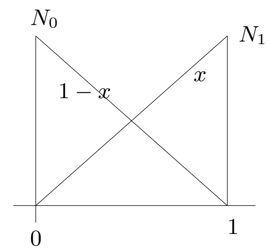
$$
f_j = \sum_{e=1} \int_{x_{e-1}}^{x_e} f(x) N_a(x) \, dx
$$

$$f_a^e = \sum_{b=1}^{2} f_b \int_{x_{e-1}}^{x_e} N_b(x) N_a(x) dx$$

Simple Examples

(1) $\mathbf{N = 1}$

(a)



$$\left.\begin{array}{ll} u'' = 0 & f \equiv 0 \\ u(0) = G & u'(1) = G \end{array}\right\} u_e = G + Hx$$

$$k'_{11} = \frac{1}{1} \cdot 1 = 1$$

$$f_1 = H \cdot 1 + \int_0^1 x \cdot 0 \, dx = H$$

$$-u_0 + u_1 = f_1$$

$$\Rightarrow u_1 = H + G$$

$$\therefore u^{h=1}(x) = GN_0(x) + (G+H)N_1(x) = G + Hx$$

(b)

$$f(x) = p$$

$$u'' + p = 0 \qquad u_p = Ax^2 \qquad 2A + p = 0$$

$$u_e(x) = \alpha + \beta x - \frac{p}{2} x^2$$

$$u(0) = G = \alpha$$

$$u'(x) = \beta - px \qquad u'(1) = \beta - p = H \qquad \beta = (p + H)$$

$$\boxed{u_e(x) = G + Hx + p\left(x - \frac{x^2}{2}\right)}$$

$$f_1 = H + \int_0^1 x \, p \, dx = H + \frac{p}{2}$$

$$k_{11} u_1 = H + \frac{p}{2} + G$$

$\Rightarrow \qquad u^h(x) = G + Hx + \dfrac{px}{2}$ does not capture exact solution because $u_e \notin \{\text{set of piecewise linear functions on } (0,1)\}$.

however $\quad u^h(0) = G \qquad u^h(1) = G + H + \dfrac{p}{2} = u_e(1)$ exact at nodes

(c)

$$f = qx$$

$$u_e(x) = G + Hx + q\frac{x}{2}\left(1 - \frac{x^2}{3}\right)$$

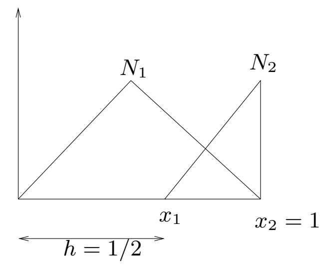
$$f_1 = H + q \int_0^1 x^2 dx = H + \frac{q}{3}$$

$$u_1 = H + \frac{q}{3} + G$$

$$
\begin{aligned}
u^h(x) &= G + Hx + \frac{qx}{3} \\
u^h(1) &= G + H + \frac{q}{3} = u_e(1) \text{ Galerkin exact at nodes}
\end{aligned}
$$

(2) $\mathbf{N = 2}$ $\quad h = \frac{1}{2}$

(a) $f \equiv 0$: $\quad u_0 = G$



$$
\frac{1}{(1/2)} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} HN_1(1) + \int_0^1 N_1 \cdot 0\, dx + 2G \\ HN_2(1) + \int_0^1 N_2 \cdot 0\, dx \end{bmatrix}
$$

$$
\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2G \\ H \end{bmatrix}
$$

$$
\begin{aligned}
u_1 &= G + \frac{H}{2} \\
u_2 &= G + H
\end{aligned}
$$

$$
\begin{aligned}
u^h(x) &= GN_0 + \left(G + \frac{H}{2}\right) N_1(x) + (G + H)N_2(x) \\
&= G + Hx
\end{aligned}
$$

(b) $\mathbf{f = p}$:

$$
\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2G + p\int_0^1 N_1\, dx \\ H + p\int_0^1 N_2\, dx \end{bmatrix} = \begin{bmatrix} 2G + p/2 \\ H + p/4 \end{bmatrix}
$$

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3p/8 + G + H/2 \\ p/2 + G + H \end{bmatrix}
$$

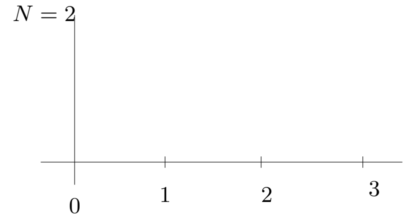
$$
\begin{aligned}
u^h(x) &= N_0 G + N_1(x)\,(3p/8 + G + H/2) + N_2(x)\,(P/2 + G + H) \\
&= G + Hx + N_1 \cdot \frac{3p}{8} + N_2 \frac{p}{2}
\end{aligned}
$$

$$
\begin{aligned}
u^h(1/2) &= G + \frac{H}{2} + \frac{3p}{8} = u_{\text{exact}}(1/2) \\
u_{\text{exact}}(x) &= (G + Hx + p(x - x^2/2)
\end{aligned}
$$

(c)

$$
\begin{aligned}
f &= qx \\
u^h &= G + Hx + \frac{11q}{48} N_1(x) + \frac{q}{3} N_2(x) \\
u^h(1/2) &= G + \frac{H}{2} + \frac{11q}{48} = u_{\text{exact}}(1/2)
\end{aligned}
$$

Using a finite difference approximation



$$
\begin{aligned}
u'' + f &= 0 \qquad & f = p \\
u(0) &= G \qquad & u'(1) = H
\end{aligned}
$$

$$
\begin{aligned}
n = 1: \qquad & \frac{U_2 - 2U_1 + G}{h^2} = -p \qquad & \frac{1}{h}[2U_1 - U_2] = hp + \frac{G}{h} \\
n = 2: \qquad & \frac{U_3 - 2U_2 + U_1}{h^2} = -p \qquad & \frac{U_3 - U_1}{2h} = H \Rightarrow U_3 = U_1 + 2hH \\
& \therefore \frac{2U_2 - 2U_1 - 2hH}{h} = hp \implies U_2 - U_1 = H + \frac{h}{2}p
\end{aligned}
$$

$$
\therefore \frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} G/h & + & hp \\ H & + & \frac{h}{2}p \end{bmatrix} \qquad h = 1/2
$$

Exactly the same equations as the FEM.

Some convergence results for the F.E. model problem:

We consider

$$
\left.\begin{aligned}
& u'' + f = 0 \\
& u(a) = G \qquad u_{,x}(b) = G
\end{aligned}\right\} (1)
$$

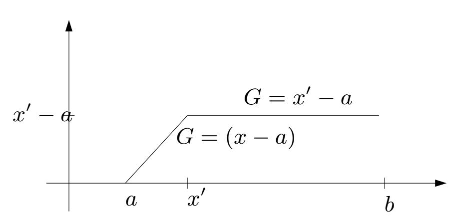
(A) **The Green's function for** (1)

$$
\text{Find} \quad G(x, x'): \quad \begin{aligned} & G_{,xx}(x, x') + \delta(x - x') = 0 \\ & G(a, x') = 0 = G_{,x}(b, x') \end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{x} \delta(x - x')dx &= \begin{cases} 0 & x < x' \\ 1 & x > x' \end{cases} \\
&= H(x - x') \\
\int_{-\infty}^{x} H(x - x')dx &= \begin{cases} 0 & x < x' \\ (x - x') & x \geq x' \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
G_{,x} + H(x - x') &= c_1 \\
G + \langle x - x' \rangle &= c_1 x + c_2 \\
\text{(i)} \qquad G_{,x}(b) + H(b - x') &= c_1 \\
0 + 1 &= c_1 \\
\text{(ii)} \qquad G(a) + \langle a - x' \rangle &= a + c_2 \\
0 + 0 &= a + c_2
\end{aligned}
$$

$$
\therefore \boxed{G(x, x') = (x - a) - \langle x - x' \rangle}
$$



Note that $G \in c^0$ is piecewise linear.

Theorem 1: $u^h(x)$ the PWL Galerkin approximation is exact at the nodes; i.e., $u^h(x_i) = u_{\text{ex}}(x_i)$

Notation: Let $v^h = \{f \in \text{ span } \{N_1, \dots, N_N\} \text{ where } N_i \in H_0'\}$.

Lemma 1: $a(\omega^h, u - u^h) = 0 \qquad \forall\, \omega^h \in v^h$

Proof:

$$\text{(W)} \;\Rightarrow\; a(u,\omega) = (f,\omega) + H\omega(b) \qquad \forall\,\omega \in H_0' \qquad (*) \qquad \begin{array}{l} u \rightarrow \text{ classical or strong solution} \\ \text{(S)} \Rightarrow \text{(W)} \end{array}$$

$$\text{(G)} \;\Rightarrow\; a(u^h,\omega^h) = (f,\omega^h) + H\omega^h(b) \qquad (**) \quad \forall \omega^h \in \begin{array}{l} v^h = \left\{\omega^h = \sum c_i N_{ij};\; N_i \in H_0'\right\} \\ v^h \subset H_0' \end{array}$$

$$\therefore (*) \;\Rightarrow\; a(u,\omega^h) = (f,\omega^h) + H\omega^h(b) \quad (***)$$

Subtract $(**)$ from $(***)$:

$$\begin{array}{ll} a(u - u^h, \omega^h) = 0 & \forall\,\omega^h \in v^h \\ a(\omega^h, u-u^h) = a(u-u^h,\omega^h) & \text{symmetry} \end{array}$$

■

Lemma 2: $a(G(x-x'),\omega) = (\delta(x-x'),\omega) \qquad \forall\,\omega \in H_0'$

Proof:

$$G_{xx} + \delta(x-x') = 0$$
$$G(a) = 0 = G_x(b)$$
$$0 = \int_0^b \left\{G_{xx} + \delta(x-x')\right\}\omega\,dx = G_x\omega\Big|_a^b + \int_a^b -G_x\omega_x + \delta\omega\,dx \qquad \forall\,\omega \in H_0'$$
$$\therefore a(G,\omega) = (\delta(x-x'),\omega) \qquad \forall\,\omega \in H_0'$$

Lemma 3: $u(x') - u^h(x') = a(G(x-x') - \omega^h, u - u^h) \qquad \forall\,\omega^h \in v^h$.

Proof: $u - u^h \in H_0'$
$u \in H_g^1$ and $u^h \in v^h + \{gN_0\} \Rightarrow u^h(a) = u(a) = g \Rightarrow u - u^h \in H_0'$

$$\begin{array}{rcl} & & \qquad\qquad \in H_0' \\ a\left(G\left(x,x'\right) - \omega^h, u - u^h\right) & = & a(G, u-u^h) - a(\omega^h, u-u^h) \\ & = & (\delta(x-x'), u-u^h) - 0 \\ & & \quad \mathcal{L}(2) \qquad\qquad \mathcal{L}(1) \\ & = & u(x') - u^h(x') \end{array}$$

■

Proof of Theorem: $u(x_i) = u^h(x_i)$

Let $x' = x_i$. Then $\mathcal{L}(3) \Rightarrow u(x_i) - u^h(x_i) = a\left(G\left(x,x_i\right) - \omega^h, u - u^h\right)$ but $x' = x_i \Rightarrow G(x,x_i) = \sum c_i N_i(x) \leftarrow$ PWL basis functions on net $\{x_0,\dots,x_N\} \Rightarrow G \in v^h \Rightarrow G - \omega^h \in v^h \overset{\mathcal{L}(1)}{\Longrightarrow} u(x_i) - u^h(x_i) = 0$ ■

Notes:

1. True no matter how large or small N is.
2. Not true for F.D. equations
3. For this problem it is as if we knew the exact solution and fed this information in at the nodes. Thus the only error in the Galerkin approximation is the interpolation error:

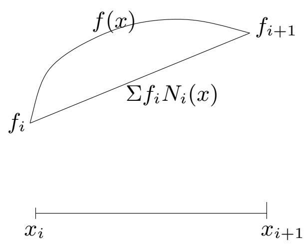
$$\therefore ||u - u^h||_\infty \leq \frac{h^2}{8}||u''||_\infty = \frac{h^2}{8}||f||_\infty \qquad \text{since } u'' + f = 0.$$

4. It is not possible to use this technique for all problems as it relied on the fact that $G \in v^h$ which is not necessarily true for all problems. For further analysis see Strang and Fix, p. 39–51.

$$||f''||_\infty = \max_{i,j} \sup_{x \in (x_i, x_j)} |u''(x)| \quad < \infty$$

Theorem: (**Error in PWL approximation**) Let $f \in PC^{2,\infty}(0,1)$ then $||f - \sum_{i=0}^{N} f_i N_i(x)|| \leq \frac{1}{8}h^2||f''||_\infty$

Proof:



for any $0 \leq i \leq N$

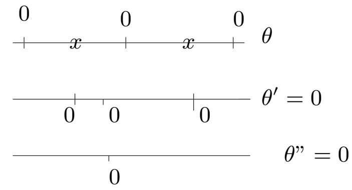
$$\begin{aligned}
\text{Let} \qquad w(x) &= (x - x_i)(x - x_{i+1}) = x^2 - (x_i + x_{i+1})x + x_i x_{i+1} \\
w' &= 2x - (x_i + x_{i+1}) = 0 \quad x = \left(\frac{x_i + x_{i+1}}{2}\right) \quad w\left(\frac{x_i + x_{i+1}}{2}\right) = \left(\frac{x_{i+1} - x_i}{2}\right)\left(\frac{x_i - x_{i+1}}{2}\right) \\
e(x) &= f(x) - \sum_{i=0}^{N} N_i(x) f_i
\end{aligned}$$

Claim: For each $x \in [x_i, x_{i+1}]$ $\exists \quad \xi_x \in [x_i, x_{i+1}] : e(x) = \frac{1}{2}f''(\xi_x)w(x)$

Proof:
1) $x = x_i$, x_{i+1} any ξ_x suffices.
2) Choose an arbitrary $x = \tilde{x}$. Choose λ such that

$$\theta(\tilde{x}) = e(\tilde{x}) - \lambda w(\tilde{x}) = 0$$

then $\theta(x)$ has 3 zeros on $[x_i, x_{i+1}]$ and
Rolle $\Rightarrow\ \exists\, \xi_x \in [x_i, x_{i+1}]\ :\ \theta''(\xi_x) = 0$



But

$$\begin{aligned}
\theta''(\xi_x) &= f''(\xi_x) - 2\lambda = 0 \\
\therefore \lambda &= \frac{1}{2} f''(\xi_x) \\
\therefore \max_{x \in [x_i, x_{i+1}]} |e(x)| &\leq \frac{1}{2} ||f''||_\infty \max_{x \in [x_i, x_{i+1}]} |w(x)| \leq \frac{1}{8} h^2 ||f''||_\infty
\end{aligned}$$

■

Theorem 2: Assume $u \in c^1[a,b]$. Then there exists at least one point $c \in (a,b)$ at which $u^h_{,x}(c) = u_{e_{x_{jx}}}(c)$.

Proof:

$$\begin{aligned}
\text{MVT} \Rightarrow \exists c \in (x_i, x_{i+1}) \quad &: \quad \underset{\displaystyle u^h(x_{i+1})}{\underset{||}{u(x_{i+1})}} = \underset{\displaystyle u^h(x_i) + (x_{i+1} - x_i) u_{,x}(c)}{\underset{||}{u(x_i)}} + (x_{i+1} - x_i) u_{,x}(c) \\
\therefore \qquad u_{,x}(c) &= \frac{u^h(x_{i+1}) - u^h(x_i)}{(x_{i+1} - x_i)} = u^h_{,x}(x)
\end{aligned}$$

$$N_i'(x) = -\frac{1}{x_{i+1} - x_i} \qquad N_{i+1}'(x) = \frac{1}{x_{i+1} - x_i} \qquad \therefore \quad u^h_{,x}(x) = \frac{u^h(x_{i+1}) - u^h(x_i)}{(x_{i+1} - x_i)}$$

The Barlow points:
If we don't know c then for linear elements $u^h_{,x}$ at the midpoints are optimally accurate:

Assume $u \in C^3(a,b)$ and expand u about $x = \alpha \in [x_i, x_{i+1}]$:

$$\begin{aligned}
u(x_{i+1}) &= u(\alpha) + (x_{i+1} - \alpha) u'(\alpha) + \frac{1}{2}(x_{i+1} - \alpha)^2 u''(\alpha) + \frac{1}{3!}(x_{i+1} - \alpha)^3 u'''(c_1) \\
u(x_i) &= u(\alpha) + (x_i - \alpha) u'(\alpha) + \frac{1}{2}(x_i - \alpha)^2 u''(\alpha) + \frac{1}{3!}(x_i - \alpha)^3 u^{(3)}(c_2)
\end{aligned}$$

Subtract and divide by $x_{i+1} - x_i = h_{i+1}$

$$\frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} = u'(\alpha) + \frac{1}{2}(x_{i+1} + x_i - 2\alpha) \frac{(x_{i+1} - x_i)}{x_{i+1} - x_i} u''(\alpha) + O(h_i^2)$$

If $\alpha = x_i$ then $u^h_{,x}(x_i) = \dfrac{u^h(x_{i+1}) - u^h(x_i)}{x_{i+1} - x_i} = u'(x_i) + \dfrac{1}{2} h_{i+1} u''(x_i) + O(h_i^2)$

If we choose $\alpha = \dfrac{x_{i+1} + x_i}{2}$ then $u^h_{,x}(x) = u'\left(\dfrac{x_{i+1} + x_i}{2}\right) + O(h_i^2)$

Conclusion:
For linear elements the *midpoints* are superconvergent with respect to derivatives, i.e. most accurate derivatives are calculated there. These are called the BARLOW points. (See Barlow, *Int. J.*

of Num. Meth. Eng., p. 243–251, 1976.

(B) **Higher order elements**:

Lagrange elements:

We use the Lagrange polynomials to construct the basis functions:

$$L_a^d(\xi) = \prod_{\substack{b=1 \\ b\neq a}}^{d+1} (\xi - \xi_b) \Big/ \prod_{\substack{b=1 \\ b\neq a}}^{d} (\xi_a - \xi_b)$$

Note: $L_a^d(\xi_c) = \delta_{ac}$ just the right properties

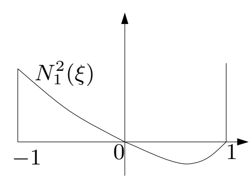
d = 1: *FIGURE*

$$L_1^1(\xi) = \frac{(\xi - 1)}{(-1-1)} = \frac{1}{2}(1-\xi) = N_1^1(\xi)$$

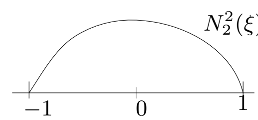
$$L_2^1(\xi) = \frac{(\xi - (-1))}{1-(-1)} = \frac{1}{2}(1+\xi) = N_2^1(\xi)$$

d = 2: *FIGURE*

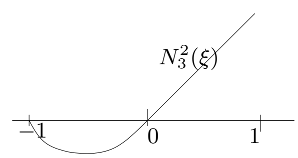
$$L_1^2(\xi) = \frac{(\xi - 0)}{(-1-0)} \, \frac{(\xi - 1)}{(-1-1)} = \frac{1}{2}\xi(\xi - 1) = N_1^2(\xi)$$



$$L_2^2(\xi) = \frac{(\xi - (-1))(\xi - 1)}{(0-(-1))(0-1)} = (1-\xi^2) = N_2^2(\xi)$$



$$L_3^2(\xi) = \frac{(\xi - (-1))\xi}{(1-(-1))(1-0)} = \frac{1}{2}\xi(1+\xi) = N_3^2(\xi)$$



Note: $\sum_{k=1}^{3} N_k^2(\xi) = \frac{1}{2}\xi^2 - \frac{1}{2}\xi + 1 - \xi^2 + \frac{1}{2}\xi + \frac{1}{2}\xi^2 = 1$

Can represent a constant function (or rigid body motion) exactly.

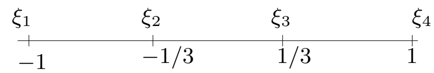
d = 3 :

$$L_1^3(\xi) = \frac{1}{16}(1-\xi)(9\xi^2-1) = N_1^3(\xi)$$

$$L_2^3(\xi) = \frac{9}{16}(3\xi-1)(\xi^2-1) = N_2^3(\xi)$$

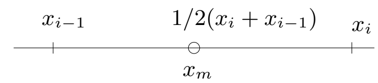
$$L_3^3(\xi) = -\frac{9}{16}(3\xi+1)(\xi^2-1) = N_3^3(\xi)$$

$$L_4^3(\xi) = \frac{1}{16}(1+\xi)(9\xi^2-1) = N_4^3(\xi)$$



Example:

$$u'' + f = 0$$
$$u(0) = G \qquad u'(1) = H$$



$$\xi = \tfrac{2}{h_i}(x - x_m)$$

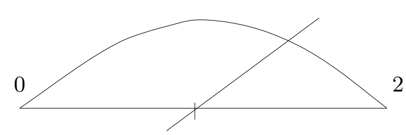
$$\tfrac{d\xi}{dx} = \tfrac{2}{h_i}$$

$$\begin{aligned} u^h &= \sum_{i=0}^{N} u_i N_i(x) \\ \mathbf{u} &= \mathbf{N}^T\mathbf{u} \\ \text{(W)} \qquad &\Rightarrow \; a(u^h, v^h) = Hv^h(b) + (f, v^h) \end{aligned}$$

$$\begin{aligned}
k^e_{ab} &= \int_{x_{e-1}}^{x_e} N'_a(x) N'_b(x)\, dx \\
&= \int_{-1}^{1} N'_a(\xi)\frac{2}{h} \cdot N'_b(\xi)\frac{2}{h} \cdot \frac{h}{2}\, d\xi \\
k^e_{11} &= \frac{2}{h}\int_{-1}^{1} \left\{\frac{1}{2}(2\xi - 1)\right\}^2 d\xi = \frac{1}{2h}\int_{-1}^{1} 4\xi^2 - 4\xi + 1\, d\xi = \frac{2}{2h}\left[\frac{4}{3}\xi^3 + \xi\Big|_0^1\right] = \frac{7}{3h} \\
k^e_{12} &= \frac{2}{h}\int_{-1}^{1} \frac{1}{2}(2\xi - 1)(-2\xi) d\xi = -\frac{2}{h}\int_{-1}^{1} 2\xi^2 - \xi\, d\xi = -\frac{2}{h} \cdot 4\frac{\xi^3}{3}\Big|_0^1 = -\frac{8}{3h} \\
k^e_{13} &= \frac{2}{h}\int_{-1}^{1} \frac{1}{2}(2\xi - 1)\frac{1}{2}(2\xi + 1) d\xi = +\frac{2}{4h} \cdot 2\int_{0}^{1} 4\xi^2 - 1 d\xi = \frac{1}{h}\left[\frac{4\xi^3}{3} - \xi\right]_0^1 = \frac{1}{3h} \\
k^e_{22} &= \frac{2}{h}\int_{-1}^{1} (-2\xi)^2 d\xi = \frac{8}{h}2\frac{\xi^3}{3}\Big|_0^1 = \frac{16}{3h} \\
[k^e] &= \frac{1}{3h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 48 & -8 & 7 \end{bmatrix}
\end{aligned}$$

Let $f = p :\Rightarrow u_e(x) = G + Hx + p\left(x - \frac{x^2}{2}\right)$

For 1 element $h = 1$



$$\begin{matrix} u_0 & u_1 & u_2 \end{matrix}$$

$$\frac{1}{3} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{8}{3}G + \frac{2}{3} \cdot p \\ H + \frac{p}{6} - \frac{G}{3} \end{bmatrix}$$

$$\begin{aligned}
f_1 &= \int_{x_{e-1}}^{x_e} p \cdot N_1(x) dx = \frac{h}{2}\int_{-1}^{1}(1 - \xi^2) d\xi = h\left[\xi - \frac{\xi^3}{3}\right]_0^1 = 2\frac{h}{3}p \\
f_2 &= H + \frac{h}{2}\int_{-1}^{1} \frac{1}{2}(\xi + \xi^2) d\xi = H + p\frac{h}{4}2\frac{\xi^3}{3}\Big|_0^1 = H + \frac{h}{6}p
\end{aligned}$$

$$\therefore \quad \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix} = \begin{bmatrix} 7/48 & 1/6 \\ \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} 8G + 2p \\ \\ -G + 3H + 1/2p \end{bmatrix} = \begin{bmatrix} 7G/6 - G/6 + 7/24p + H/2 + 1/12p \\ \\ 4G/3 + 1/3p - G/3 + H + p/16 \end{bmatrix}$$

$$= \begin{bmatrix} G + 3/8p + H/2 \\ \\ G + 1/2p + H \end{bmatrix} \qquad \begin{matrix} N_1(x) = 4x(1-x) \\ N_2(x) = x(2x-1) \end{matrix}$$

$$\begin{aligned}
\therefore u^h(x) &= GN_0(x) + \left(G + H/2 + \frac{3}{8}p\right) N_1(x) + (G + p/2 + H)\, N_2(x) \\
&= G + (H/2N_1\,(x) + HN_2\,(x)) + \left(\frac{3}{8}pN_1\,(x) + \frac{p}{2}N_2\,(x)\right) \\
&= G + H\left(\frac{1}{2}\left(4x - 4x^2\right) + 2x^2 - x\right) + p\left(\frac{3}{8}\left(4x - 4x^2\right) + \frac{1}{2}(2x^2 - x)\right) \\
&= G + Hx + \frac{p}{2}\left\{3x - 3x^2 + 2x^2 - x\right\} \\
&= G + Hx + \frac{p}{2}\left\{2x - x^2\right\} \\
&= G + Hx + p\left(x - \frac{x^2}{2}\right) \qquad \text{the exact solution}
\end{aligned}$$