2. Introduction to PDE

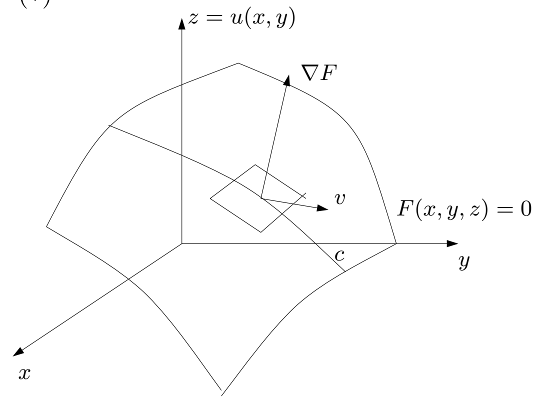
2.1 Classification of PDE

1st order PDE

$$\begin{aligned} F(x,y,u_x,u_y) &= 0 \\ \text{Eg:} \quad uu_x + u_t &= 1 \quad \text{shock waves in traffic flow and fluid mechanics} \end{aligned}$$

Solving 1st order PDE using the method of characteristics

$$\boxed{a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)} \qquad (*)$$



The solution $z = u(x,y)$ is a surface.
Now consider the surface

$$\begin{aligned} F(x,y,z) &= u(x,y) - z = 0 \\ \text{Then } \boldsymbol{\nabla} F &= (u_x, u_y, -1) \text{ is a normal to the} \\ \text{surface} \quad F &= 0. \end{aligned}$$

Now the PDE $(*)$ can be rewritten in the form

$$\mathbf{v} \cdot \boldsymbol{\nabla} F = (a,b,c) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial y}{\partial y}, -1\right) = 0.$$

Thus $\mathbf{v} = (a,b,c)$ represents a tangent vector to the solution surface $F = 0$ at the point $(x, y, z = u)$.

We can construct a curve $C : (x(t), y(t), z(t))$ lies in the solution surface for which $\mathbf{v}$ is a tangent at each point. Since $\mathbf{v}$ is tangent to c it follows that the tangent vector to c is

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \Big|\Big| (a,b,c) \Leftrightarrow \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \alpha(a,b,c) \quad \text{or equivalently}$$

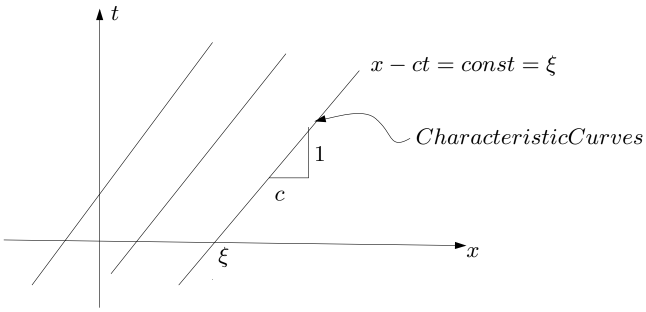
$$\boxed{\frac{dx}{dt} = a(x,y,u) \quad \frac{dy}{dt} = b(x,y,u) \quad \frac{du}{dt} = c(x,y,u)} \qquad (**)$$

by defining the arclength of C to be such that $\alpha = 1$. Thus the solution of the PDE $(*)$ has been reduced to solving the system of ODE's.

Eg. 1: 1D wave equation $u_t + cu_x = 0; \quad u(x,0) = f(x)$.

$$\frac{dx}{dt} = \frac{c}{1}$$

$$\begin{aligned} \Rightarrow x - ct &= \text{constant } = \xi \quad \frac{du}{d\xi} = 0 \Rightarrow u = \text{ const } = B \\ u(x,0) &= f(x) = B \qquad x - c \cdot 0 = \xi \Rightarrow x = \xi \Rightarrow B = f(\xi) \\ \therefore u(x,t) &= f(x - ct). \end{aligned}$$



Cauchy problem

Given $u(x, y)$ along $C : y = y(x)$, when can we determine u_x and u_y?

$$\begin{aligned} u(x, y(x)) &= f(x) \\ u_x + u_y y' = f'(x) & \\ a u_x + b u_y &= c \\ \begin{bmatrix} 1 & y' \\ a & b \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} &= \begin{bmatrix} f' \\ c \end{bmatrix} \end{aligned}$$

Cannot calculate u_x and u_y when

$$\begin{aligned} \det\left(\begin{pmatrix} 1 & y' \\ a & b \end{pmatrix}\right) &= 0 \\ \text{or} \qquad b - a y' &= 0 \\ b &= a\frac{dy}{dx} \end{aligned}$$

$$\boxed{\frac{dx}{a} = \frac{dy}{b}}$$

Cannot specify data along a characteristic curve.

2nd order PDE $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ $\qquad (*)$

- Higher order PDE often occur, but we already have an extremely rich class of PDE in $(*)$.
- Linear if F is linear in each term involving u.
- Quasilinear if linear in the highest derivatives.

Eg:

Heat Eq:	$u_t = u_{xx}$	
Wave Eq:	$u_{tt} = c^2 u_{xx}$	
Laplace's Eq:	$u_{xx} + u_{yy} = 0$	
Burger's Eq:	$u_t + u u_x = u_{xx}$	quasilinear, shocks smoothed by viscosity
Porous Media Eq:	$u_t = (\beta(u) u_x)_x$	nonlinear

Classification of general 2nd order linear PDE

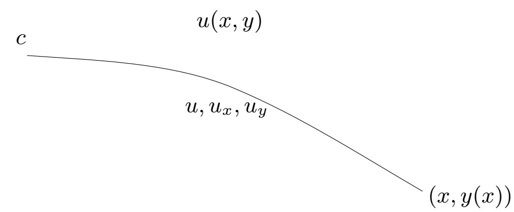
$$Lu = \underbrace{au_{xx} + bu_{xy} + cu_{yy}}_{\text{Principal part}} + \underbrace{du_x + eu_y + fu}_{\text{Lower order terms}} = \underbrace{g}_{\text{Inhomogeneous term}} \qquad a = a(x, y) \text{ variable ?}$$

by analogy with quadratic forms $aX^2 + bXY + cY^2 + \ldots$ we define the equations to be

(a) Hyperbolic if $\quad b^2 - 4ac > 0$
(b) Parabolic if $\quad b^2 - 4ac = 0$
(c) Elliptic if $\quad b^2 - 4ac < 0.$

Cauchy Problem

If we are given u, u_x, u_y along some curve $c : y = y(x)$, i.e., $u(x, y(x)) = F(x)$, $\quad u_x(x, y(x)) = G(x)$, $\quad u_y(x, y(x)) = H(x)$. Can we determine $u(x, y)$ at some neighboring point?



$$\begin{aligned} &u_{xx} + u_{xy}y' = G'(x) \\ &u_{xy} + u_{yy}y' = H'(x) \\ &a\, u_{xx} + b\, u_{xy} + c\, u_{yy} = g - \text{LOT} = K(x) \end{aligned}$$

$$\begin{bmatrix} 1 & y' & 0 \\ 0 & 1 & y' \\ a & b & c \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} G' \\ H' \\ K \end{bmatrix}$$

We can calculate $u_{xx}, u_{x,y}$ and u_{yy} provided $\det(\cdot) = a(y')^2 - by' + c \neq 0$

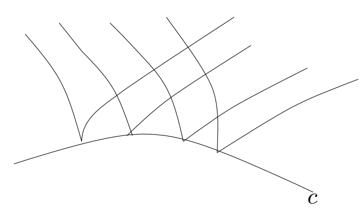
(i) If $b^2 - 4ac > 0$ we get 2 curves along which data cannot be specified and used to get a neighboring solution. These curves are called characteristics and are defined by $y' = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$.

Eg: Wave equation

$$\left(\frac{\partial z}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0$$

$$b^2 - 4ac = 0^2 - 4(1)(-c^2) = 4c^2 > 0$$

$x \pm ct =$ const are characteristics



(ii) If $b^2 - 4ac = 0$ we get 1 characteristic curve

Eg.

$$u_t - u_{xx} = 0$$

$$b^2 - 4ac = 0 - 4(-1)(0) = 0$$

$$\frac{dt}{dx} = 0 \text{ is a characteristic}$$

$t =$ const are characteristics

(Signals propagate with infinite speed.)

(iii) If $b^2 - 4ac < 0$ there are no characteristic curves

Eg: Laplace's equation $u_{xx} + u_{yy} = 0$

$$b^2 - 4ac = 0 - 4 = -4 < 0$$

Thus, Cauchy data can be specified for any curve to obtain a neighboring solution. This presents a problem if Cauchy data are specified for a boundary value problem – over specified.

$$\boxed{u_{xx} + u_{yy} = 0}$$

Prototype parabolic problem

$$u_t = \underbrace{Du_{xx}}_{\text{Diffusion}} - \underbrace{cu_x}_{\text{Convection}} - \underbrace{bu}_{\text{Cooling}} + \underbrace{f(x,t)}_{\text{External input/output of heat}} \qquad x \in \Omega$$

$D > 0$ – Diffusion coefficient

c – Wave speed ($c > 0 \Rightarrow$ wave moves in positive x direction)

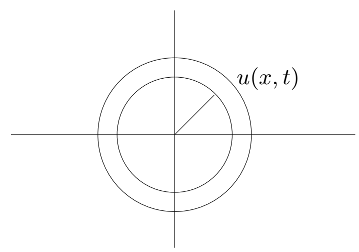
b – heat transfer coefficient ($b > 0$ heat loss, $b < 0$ heat gain)

Later we will consider the cases $b(x)$ – variable coefficients
$b(x, u)$ – quasilinear

Different types of boundary conditions:

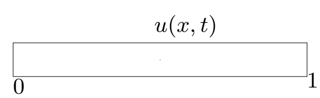
1) Periodic BC – temperature in a conducting ring

- $\Omega = (0, 2\pi)$
- 'Boundary Condition': $u(x,t) = u(x + 2\pi, t)$ – u is periodic
- Initial Condition $u(x, 0) = u_0(x)$
- For the solution to make physical sense $b > 0$ otherwise $u \Rightarrow \infty$.



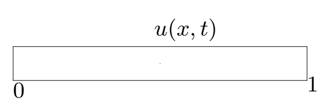
2) Dirichlet BC – temperature in a bar with fixed end temperature

- $\Omega = (0, 1)$
- Dirichlet BC – $u(0,t) = \alpha(t) \qquad u(1,t) = \beta(t)$
- Initial Condition: $u(x, 0) = u_0(x)$.



3) Mixed BC – Temperature in a bar with one end at a specified temperature and the other at a specified flux.

- $\Omega = (0, 1)$
- Mixed BC $\quad u(0,t) = \alpha(t), \quad \dfrac{\partial u}{\partial x}(1,t) = \beta(t)$



Time Independent Problem

- Will return to the parabolic problem later.
- Assume f, α and β do not depend on time. Then we can show that $u(x,t) \overset{t\to\infty}{\longrightarrow} u(x)$ a steady state.

$u(x)$ satisfies the steady state equation:

$$\boxed{\begin{array}{l} D\ u_{xx} - cu_x - bu = f(x) \\ \bullet\ u \text{ satisfies periodic, Dirichlet or mixed BC.} \end{array}}$$

This is our prototype elliptic problem

- Elliptic problems arise in
  – Steady state for problems with diffusion or viscosity.
  – Potential problems.
- Mathematical characterization of elliptic problems.
  – Unique solutions that are smoother (i.e., have more derivatives) than the data function f.
- Why is this a prototype problem?
  – Only 1 space dimension – problem character does not change in 2D or 3D but there are extra numerical issues that arise (e.g. iterative solution methods, boundary conditions).
  – For periodic problem there are no boundary conditions, which makes the analysis easier.

Discretization Process:

- Periodic case: $\Omega = [0, 2\pi]$
  Divide domain into N sample points

$x_k = (2\pi/N)k$ $\quad k = 0, 1, ..., N-1$

$h = (2\pi)/N$

- Dirichlet and mixed cases: $\Omega = (0, 1)$
  Divide domain into $N + 1$ sample points

$x_k = (1/N)k \quad k = 0, ..., N \quad h = 1/N$

Sample u at each of the grid points with a uniform spacing h. We use capital letters to denote approximate values at grid points:

$$\begin{array}{rcl} U_k & \simeq & u(x_k) \\ F_k & = & f(x_k) \\ C_k & = & c(x_k) \\ B_k & = & b(x_k) \end{array} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{Exact}$$

We will consider the following types of discretizations for the prototype problem.

(I) Finite Difference

(II) Spectral – For the periodic case

(III) The method of weighted residuals

- Collocation
- Galerkin

(IV) The finite element method

The finite difference method

Idea: Approximate derivatives by difference quotients.

Periodic Problem: $-D\,u_{xx} + bu = f, \qquad u(x+2\pi) - u(x), \qquad D = 1$

$$u_{xx} \simeq \frac{\delta^2 U_n}{h^2} = \frac{U_{n+1} - 2U_n + U_{n-1}}{h^2}$$

$$\therefore \boxed{\frac{-1}{h^2}U_{n+1} + \left(\frac{2}{h^2} + B_n\right)U_n - \frac{U_{n-1}}{h^2} = F_n} \qquad n = 0, \ldots, N-1$$

Periodicity $\Rightarrow U_N = U_0 \qquad U_{N-1} = U_{-1}$

so we have the matrix problem

$$A^h U = F^h$$

$$\text{where} \qquad A^h = \begin{bmatrix} (2/h^2 + B_0) & -1/h^2 & 0\ldots0 & -1/h^2 \\ -1/h^2 & (2/h^2 + B_1) & -1/h^2 & 0\ldots0 \\ & 0 & \ddots & \\ \vdots & & & \\ 0 & & & \\ -1/h^2 & 0\ldots0 & -1/h^2 & (2/h^2 + B_{N-1}) \end{bmatrix}$$

Properties of A:

- A is symmetric
- A is positive definite
- A is diagonally dominant i.e., $|A_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{N} |A_{ij}|$
- A is almost tridiagonal – can be solved in $O(N)$ operations

Questions:

(1) Is $AU = F$ solvable? Yes, all eigenvalues are positive.

(2) How close is U to u?
We expect $||U - u|| \leq kh^2$ but we need to do some work to prove this.

Truncation Error: The truncation error (T.E.) is the remainder you get when you substitute the exact solution to $-Du_{xx} + bu = f(*)$ into the difference equation.

$$\text{i.e.:} \qquad T_h = -\frac{\delta^2}{h^2}u_i + B_i u_i - F_i = O(h^2).$$

A difference scheme is *consistent* with the differential equation $(*)$ if $T_h \Rightarrow 0$ as $h \Rightarrow 0$.

Vector and Matrix Norms

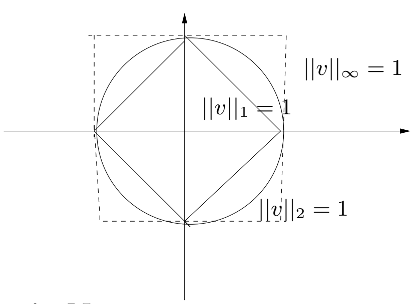
Vector Norms: Let $\mathbf{x} \in \mathbb{R}^N$, then $||\cdot|| : \mathbb{R}^N \to \mathbb{R}^+$ is a real valued function satisfying:

$$\begin{array}{lll} \text{(i)} & ||x|| \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^N & ||x|| = 0 \Leftrightarrow \mathbf{x} = 0 \\ \text{(ii)} & ||c\mathbf{x}|| = |c|\,||\mathbf{x}|| \quad \forall \mathbf{x} \in \mathbb{R}^N,\ c \in \mathbb{R} & \\ \text{(iii)} & ||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^N & \Delta \text{ inequality.} \end{array}$$

If $||\cdot||$ satisfies (i)–(iii) then it is called a vector norm.

Examples:

$$\begin{array}{rll} ||\mathbf{x}||_1 = & \sum_{i=1}^N |x_i| & \text{absolute sum norm} \\ ||\mathbf{x}||_2 = & \left(\sum_{i=1}^N |x_i|^2\right)^{1/2} & \text{Euclidean norm} \\ ||\mathbf{x}||_\infty = & \max_i |x_i| & \text{maximum norm} \end{array}$$



The sets of points in $\mathbb{R}^2$ for which the various norms are 1 i.e. unit circles.

Matrix Norms:

A matrix norm is a function $||\cdot|| : \mathbb{R}^N \times \mathbb{R}^N \Rightarrow \mathbb{R}^+$ which satisfies the properties

$$\begin{array}{ll} \text{(i)} & ||A|| > 0 \qquad ||A|| = 0 \Leftrightarrow A \equiv 0 \\ \text{(ii)} & ||cA|| = |c|\,||A|| \\ \text{(iii)} & ||A+B|| \leq ||A|| + ||B|| \end{array}$$

A matrix norm with the property $||AB|| \leq ||A||\,||B||$ is called *multiplicative*.

A matrix norm and a vector norm are *consistent* if

$$||Ax|| \leq ||A||\,||x|| \qquad ||x|| \neq 0 \Rightarrow \frac{||Ax||}{||x||} \leq ||A||.$$

Induced matrix norms:

Define $||A|| = \max_{||x||\neq 0} \frac{||Ax||}{||x||} = \max_{||x||=1} ||Ax||$ – Lengths of images of the unit sphere.

Example 1. $||A||_\infty =$ max row sum of the elements of $A = \max_i \sum_j |a_{ij}|$

Proof:

$$\begin{aligned} ||x||_\infty &= \max_i |x_i| \\ ||Ax||_\infty &= \max_i \left|\sum_j a_{ij}x_j\right| \overset{\Delta\text{ineq}}{\leq} \max_i \sum_j |a_{ij}x_j| \leq \left(\max_i \sum_j |a_{ij}|\right) ||x||_\infty \\ \therefore \frac{||Ax||_\infty}{||x||_\infty} &\leq \left(\max_i \sum_j |a_{ij}|\right) \ \therefore\ ||A||_\infty \leq \max_i \sum_j |a_{ij}| \qquad (*) \end{aligned}$$

If $\max_i \sum_j |a_{ij}| = \sum_j |a_{kj}|$ for some row index k, then let

$$\hat{x} = (\bar{a}_{k1}/|a_{k1}|, \ldots, \bar{a}_{kN}/|a_{kN}|) \Rightarrow \sum_j a_{kj}\hat{x}_j = \sum_j |a_{kj}|^2/|a_{kj}| = \sum_j |a_{kj}|.$$

If for some index j, $a_{kj} = 0$ then let $\hat{x}_j = 1$. Then $||\hat{x}||_\infty = 1$, and

$$\begin{aligned} ||A\hat{x}||_\infty &= \max_i \left|\sum_j a_{ij}\hat{x}_j\right| \geq \sum_j |a_{kj}| = \sum_j |a_{kj}|\, ||\hat{x}||_\infty = \max_i \sum_j |a_{ij}|\, ||\hat{x}||_\infty \\ ||A||_\infty &\geq \frac{||A\hat{x}||_\infty}{||\hat{x}||_\infty} \geq \max_i \sum_j |a_{ij}| \qquad (**) \end{aligned}$$

Combining $(*)$ and $(**)$ we have $||A||_\infty = \max_i \sum_j |a_{ij}|$

Exercise 2: $||A||_1 =$ maximum column sum of the moduli of elements of $A = \max_j \sum_i |a_{ij}|$

Example 3: $||A||_2 =$ (maximum eigenvalue of $A^*A)^{1/2} = \rho(A^*A)$ where $\rho(B) = \max_j |\lambda_j|$ where λ_j are the eigenvalues of B, is known as the spectral radius of B.

Proof: Since A^*A is Hermitian there exists a unitary matrix u (for which $u^*u = I$) such that

$$u^*(A^*A)u = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_N \end{bmatrix}$$

where $\mu_i \geq 0$ are the eigenvalues of A^*A. Let $y = u^*x$ so that $x = uy$. Then

$$\begin{aligned} ||A||_2 = \max_{||x||\neq 0} \frac{||Ax||_2}{||A||_2} &= \max_{||x||\neq 0} \sqrt{\frac{\langle A^*Ax, x\rangle}{\langle x, x\rangle}} \qquad ||Ax|| = (Ax)^*(Ax) = (A^*Ax)(x) \\ &= \max_{||y||\neq 0} \sqrt{\frac{\langle u^*A^*Auy, y\rangle}{\langle u^*uy, y\rangle}} \end{aligned}$$

$$
\begin{aligned}
&= \max_{||y|| \neq 0} \sqrt{\frac{\sum_i \mu_i |y_i|^2}{\sum |y_i|^2}} \\
&= \sqrt{\max |\mu_i|} \\
\therefore ||A||_2 &= \rho(A^* A)
\end{aligned}
$$

■

Note: If A is symmetric $||A||_2 = \max_i |\lambda_i|$. Also $||A^{-1}||_2 = \dfrac{1}{\min |\lambda_i|}$.

Error estimate for the finite difference method:

Let us look at the size of the error $e = u - U$.

$$AU = F \tag{1}$$
$$Au = F + T_h \tag{2}$$

where T_h is the truncation error and $||T_h||_\infty = O(h^2)$ and $||T_h||_2 = O(h^2)$. Subtract (1) from (2):

$$
\begin{aligned}
Ae &= T_h \\
e &= A^{-1} T_h.
\end{aligned}
$$

We want $||e||_?$ to be $O(h^2)$ the same as T_h, so we must have that $||A^{-1}||_?$ is bounded independent of h.

Definition: (Norm stability)

A discretization $A^h u^h = F^h$ for any elliptic problem

ℓ_∞ : is said to be *max-norm stable* if

$$||(A^h)^{-1}||_\infty \le K \quad \text{for all } h.$$

ℓ_2 : is said to be ℓ_2 norm stable if

$$||(A^h)^{-1}||_2 \le K \qquad \text{for all } h.$$

Convergence Theorem:

A consistent, stable discretization for a linear elliptic problem converges with the order of the truncation error:

PF: $\quad ||e||_\infty \le k||T_h||_\infty \qquad ||e||_2 \le k||T_h||_2.$

Claim 1: The finite difference matrix for the periodic problem with constant heat transfer coefficient $B_n = B$:

$$A.^h = -\frac{E}{h^2} + \left(\frac{2}{h^2} + B\right) I - \frac{E^{-1}}{h^2}$$

is ℓ_2 -norm stable.

Observe that the DFT basis vectors $\phi_j^k = e^{i\frac{2\pi}{N}jk} \quad k = 0, 1, \ldots, N-1$ are eigenvectors of A^h

$$\begin{aligned}
A.^h\phi_j^k &= -\frac{e^{i\left(\frac{2\pi}{N}\right)k(j+1)h}}{h^2} + \left(\frac{2}{h^2} + B\right) e^{i\left(\frac{2\pi}{N}\right)kjh} - \frac{e^{i\left(\frac{2\pi}{N}\right)k(j-1)h}}{h^2} \\
&= \left\{\frac{2 - 2\cos(kh\,\pi/N)}{h^2} + B\right\}\phi_j^k \\
&= \left\{\frac{4\sin^2(kh\,\pi/2N)}{h^2} + B\right\}\phi_j^k \\
&= \lambda^k\phi_j^k
\end{aligned}$$

Note:

- Eigenvalues λ^k are all positive.
- $||A^{-1}||_2 = \dfrac{1}{\min|\lambda^k|} = \dfrac{1}{B}$ which is bounded independent of h.
- The fact that the DFT basis vectors ϕ_j^k diagonalize A^h can be used as a computational device to invert the matrix A^h. Let $\hat{u}^k = FFT(U)$ and $\hat{F}^k = FFT(F)$. Then since $A^h\phi^k = \lambda^k\phi^k$ and $U = \sum \hat{u}^k\phi^k$, $F = \sum \hat{F}^k\phi^k$. It follows that $\lambda^k\hat{U}^k = \hat{F}^k$.

  $$\therefore \hat{U}^k = \hat{F}^k/\lambda^k$$

  so that $u = FFT^{-1}\left(\hat{U}^k\right)$.
- The above analysis and inversion technique only works for constant coefficients b. It is possible to analyze the stability of a variable coefficient problem by freezing coefficients and performing a DFT stability analysis.

The Dirichlet Problem:

$$\begin{aligned}
y'' &= f(x, y, y') \\
y(0) &= \alpha \qquad y(1) = \beta.
\end{aligned}$$

$$y_{n+1} - 2y_n + y_{n-1} = h^2 f\left(x_n, y_n, \frac{y_{n+1} - y_{n-1}}{2h}\right) = h^2 f_n \tag{3}$$

$$y_0 = \alpha \qquad\qquad y_N = \beta$$

$\downarrow$ from B.-C.

$$\begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 \\
1 & -2 & 1 & \cdots & 0 \\
 & & \ddots & & \vdots \\
 & \ddots & & & 0 \\
 & & \ddots & & 1 \\
0 & & & & \\
 & & & & 1 - 2
\end{bmatrix}
\begin{bmatrix} y_1 \\ \\ \vdots \\ \\ \vdots \\ \\ y_{N-1} \end{bmatrix}
= h^2 \begin{bmatrix} f_1 \\ \\ \vdots \\ \\ \vdots \\ \\ f_{N-1} \end{bmatrix}
- \begin{bmatrix} \alpha \\ 0 \\ \\ \vdots \\ \\ 0 \\ \beta \end{bmatrix}$$

Tridiagonal $\quad A\mathbf{y} = h^2\mathbf{f}(\mathbf{y}) - \mathbf{r}$

$$0 = \mathbf{g}(\mathbf{y}^{k+1}) = \mathbf{g}(\mathbf{y}^k) + \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{y}^k)(\mathbf{y}^{k+1} - \mathbf{y}^k)$$

$$\therefore \boxed{\mathbf{y}^{k+1} = \mathbf{y}^k - \left[\frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{y}^k)\right]^{-1} \mathbf{g}(\mathbf{y}^k)}$$

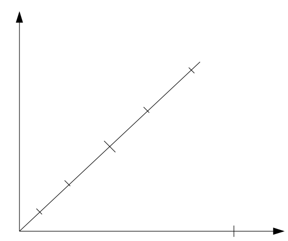
Solve using Newton Iteration

$$\mathbf{g}(\mathbf{y}) = A\mathbf{y} - h^2\mathbf{f}(\mathbf{y}) + \mathbf{r} = 0$$

$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} - \left[\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\left(\mathbf{y}^{(k)}\right)\right]^{-1} \mathbf{g}\left(\mathbf{y}^{(k)}\right)$$

Eg. 1 $y'' = 0 \qquad y(0) = 0 \qquad y(1) = 1 \Rightarrow \boxed{y(x) = x}$

$$\begin{aligned}
y_{n+1} &- 2y_n + y_{n-1} = 0 \qquad 1 \le n \le N-1 \\
y_n &= \theta^n \Rightarrow \theta^2 - 2\theta + 1 = 0 \\
& \quad \theta = 1, 1 \\
y_n &= A + Bn \\
y_0 &= A = 0 \\
y_N &= BN = 1 \Rightarrow \boxed{y_n = \left(\tfrac{n}{N}\right) = nh = x_n}
\end{aligned}$$

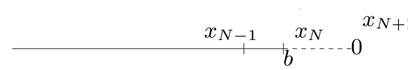


- Shape of solution was captured exactly by the quadratic variation assumed by the difference approximation.

Special Tricks:

(1) **For derivative boundary conditions**:

$$y'(b) = \beta \text{ say}$$



we introduce the pseudo meshpoint x_{N+1} and we have the condition

$$\frac{y_{N+1} - y_{N-1}}{2h} = \beta \Longrightarrow y_{N+1} = (y_{N-1} + 2h\beta)$$

Let's look at the effect on the simple problem $y'' = 0 \qquad y(a) = \alpha \qquad y'(b) = \beta$

$$\begin{matrix} y_1 & y_2 \end{matrix}$$

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & & & & \\ & \ddots & & \ddots & \\ & & 1 & & \\ & & 2 & & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ \\ \\ \\ y_N \end{bmatrix} = - \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ 2h\beta \end{bmatrix}$$

(2) For self-adjoint problems we often have: $(p(x)y')' + q(x)y = T(x)$. In this case we use

$$\frac{1}{h}\left[p_{n+1/2}\left(\frac{y_{n+1} - y_n}{h}\right) - p_{n-1/2}\left(\frac{y_n - y_{n-1}}{h}\right)\right].$$

Eg. 1 with derivative BC:

$$
\begin{aligned}
y'' &= x \qquad y(0)=0 \qquad y'(1)=0 \\
y &= \frac{x^3}{6} + Ax + B \qquad y(0) = B = 0 \\
y'(x) &= \frac{x^2}{2} + A \Rightarrow y'(1) = \frac{1}{2} + A = 0 \Rightarrow A = -\frac{1}{2}
\end{aligned}
$$

$$\therefore \boxed{y(x) = \frac{x^3}{6} - \frac{x}{2}.}$$

Homog. eq.

$$
\begin{aligned}
&y_{n+1} - 2y_n + y_{n-1} = 0 \qquad O(h^2) \\
&y_n = \theta^n \Rightarrow (\theta-1)^2 = 0 \Rightarrow \theta = 1, 1 \\
&y_n = an + b
\end{aligned}
$$

Particular solution

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} &= h^2 \overset{\overset{x_n}{\|}}{(nh)} = h^3 n \\
y_n = cn^3 &\Rightarrow c\left[(n+1)^3 - 2n^3 + (n-1)^3\right] \\
&= c\left[n^3 + 3n^2 + 3n + 1 - 2n^3 + n^2 - 3n^2 + 3n - 1\right] \\
&= 6nc = h^3 n \\
&\therefore \; c = \frac{h^3}{6} \\
\therefore y_n = \frac{n^3h^3}{6} + an + b &= \frac{(nh)^3}{6} + an + b.
\end{aligned}
$$

$$y_0 = 0 \rightarrow b = 0.$$

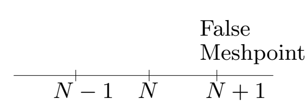
BC 1:

$$\frac{y_N - y_{N-1}}{h} = 0 \qquad O(h)$$

$$
\begin{aligned}
0 &= \frac{N^3h^3}{6} + aN - \left[\frac{(N-1)^3h^3}{6} + a(N-1)\right] \Rightarrow a\left[N - (N-1)\right] = \frac{-h^3}{6}\left[N^3 - (N-1)^3\right] \\
\therefore a &= -\frac{h^3}{6}\left[N^3 - N^3 + 3N^2 - 3N + 1\right] = -\frac{h^3}{6}\left[3N^2 - 3N + 1\right] \qquad hN = 1 \\
y_n &= \frac{x_n^3}{6} - \frac{nh^3}{6}\left[3N^2 - 3N + 1\right] = \frac{x_n^3}{6} - \frac{x_n}{2}\left[h^2N^2 - h(hN) + \frac{1}{3}h^2\right] \\
&= \frac{x_n^3}{6} - \frac{x_n}{2} + \frac{x_n}{2}\left(h - \frac{h^2}{3}\right) \Leftarrow O(h) \longrightarrow \text{(comes from BC)}
\end{aligned}
$$

BC 2:

$$
\begin{aligned}
&\frac{y_{N+1} - y_{N-1}}{2h} = 0 \Rightarrow y_{N+1} = y_{N-1} \\
&\frac{(N+1)^3h^3}{6} + a(N+1) = \frac{(N-1)^3h^3}{6} + a(N-1)
\end{aligned}
$$



$$\begin{aligned}
\therefore 2a &= \frac{h^3}{6}\left[(N-1)^3-(N+1)^3\right] \\
&= -\frac{h^3}{6}\left[N^3+3N^2+3N+1-N^3+3N^2-3N+1\right] \\
a &= -\frac{h^3}{6}\left[3N^2+1\right] \\
\therefore y_n &= \frac{x_n^3}{6}-\frac{(nh)h^2}{6}\left(3N^2+1\right) \\
&= \underbrace{\frac{x_n^3}{6}-\frac{x_n}{2}}_{\text{exact}}-\underbrace{\frac{x_n h^2}{6}}_{\text{error}} \quad \longrightarrow \quad O(h^2)
\end{aligned}$$

Eg:

$$\begin{aligned}
&y''+4y=0 \\
&y(0)=0 \quad y(1)=1 \\
&y=A\sin 2x+B\cos 2x \\
&y(0)=0\Rightarrow B=0 \\
&y(1)=1\Rightarrow A\sin 2=1\Rightarrow A=\frac{1}{\sin 2} \\
&\therefore y(x)=\frac{\sin 2x}{\sin 2}
\end{aligned}$$

$$y_{n+1}-2y_n+y_{n-1}+4h^2y_n=0$$

$$\begin{aligned}
y_n=\theta^n \qquad & \theta^2-(2-4h^2)\theta+1=0 \\
\theta_1\theta_2=1 \qquad & \theta=e^{i\alpha} \\
& e^{i\alpha}-(2-4h^2)+e^{-i\alpha}=0 \\
& 2(1-\cos\alpha)=4h^2 \qquad \cos\alpha=1-2\sin^2\alpha/2 \\
& 4\sin^2\alpha/2=4h^2 \\
& \therefore \sin^2\alpha/2=h^2 \qquad \alpha=2\sin^{-1}h \\
& y_n=A\cos\alpha n+B\sin\alpha n \\
& y_0=0\Rightarrow A=0 \qquad y_N=B\sin\alpha N=1 \\
\therefore \qquad & B=\frac{1}{\sin(\alpha N)} \\
\therefore \qquad & y_n=\frac{\sin(2n\sin^{-1}h)}{\sin(2N\sin^{-1}h)} \qquad \sin^{-1}h=h+\frac{h^3}{6}+O(h^5).
\end{aligned}$$

Eg. 2: An eigenvalue problem

$$\begin{aligned}
y''+\lambda^2 y &= 0 \\
y(0) &= 0=y(1) \\
y &= A\cos\lambda x+B\sin\lambda x \\
y(0) &= A=0\Rightarrow y(x)=B\sin\lambda x
\end{aligned}$$

$$y(1) = 0 = B\sin\lambda \qquad \Rightarrow \begin{array}{l}\lambda = n\pi \text{ for nontrivial sol.}\\ n = 1, 2\ldots\end{array}$$

$$\Rightarrow \boxed{y_k(x) = B\sin(k\pi x)\,;\lambda = k\pi}$$

$$FD \Rightarrow \quad \begin{array}{l} y_{n+1} - 2y_n + y_{n-1} + h^2\lambda^2 y_n = 0 \qquad n = 1,\ldots,N-1 \\ y_{n+1} - (2-r^2)y_n + y_{n-1} = 0 \ \leftarrow \text{ discrete eigenvalue problem } \ r = (h\lambda) \\ y_n = \theta^n : \theta - (2-r^2) + \theta^{-1} = 0 \qquad\qquad \theta_1\theta_2 = 1 \end{array}$$

$$\theta = e^{i\alpha}$$

$$2[\cos\alpha - 1] + r^2 = 0$$

$$\boxed{h^2\lambda^2 = 4\sin^2\left(\frac{\alpha}{2}\right)}$$

Aside: If we had tried: $\theta = e^{\alpha}$

$$e^{\alpha} - (2-r^2) - e^{-\alpha} = 0$$

$$2\text{cosh}\alpha - 2 - r^2 = 0$$

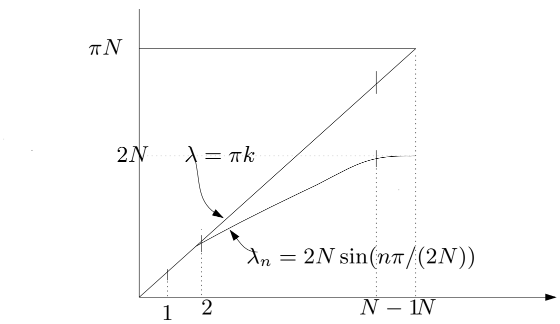
$$4\text{sinh}^2\frac{\alpha}{2} + r^2 = 0$$

roots unless $\alpha \in \mathbb{C}$

Recall $\cos\alpha - 1 = 2\sin^2\left(\frac{\alpha}{2}\right)$

$$\begin{aligned} y_n &= A\cos(\alpha n) + B\sin(\alpha n) \\ y_0 &= A = 0 \\ y_N &= B\sin(\alpha N) = 0 \Rightarrow \alpha = \frac{k\pi}{N} \qquad k = 1,\ldots,N-1 \end{aligned}$$

$$\boxed{y_{k,n} = B_k\sin\left(\frac{k\pi n}{N}\right); \quad \lambda_k = \frac{2}{h}\sin\left(\frac{k\pi}{2N}\right) = 2N\sin\left(\frac{k\pi}{2N}\right)}$$



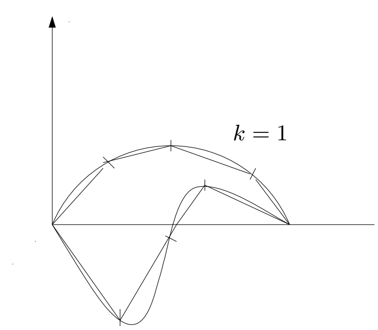
$N \gg 1 : \quad k = 1$

$$\lambda_1 = 2N\sin\left(\frac{\pi}{2N}\right) \approx 2N\cdot\frac{\pi}{2N} = \pi$$

Asymptotic behavior of eigenvalues:

$$\begin{aligned} \lambda_k &= 2N\left\{\frac{k\pi}{2N} - \frac{1}{6}\left(\frac{k\pi}{2N}\right)^3 + \ldots\right\} \\ &= k\pi - O\left(\frac{1}{N^2}\right) \qquad k \ll N \\ \lambda_k(N) &= 2N\sin\left(\frac{k\pi}{2N}\right) = \frac{2\sin\left(\frac{k\pi}{2N}\right)}{(1/N)} \overset{N\to\infty}{\longrightarrow} \frac{2\cos\left(\frac{k\pi}{2N}\right)\cdot\frac{k\pi}{2}}{1} \Rightarrow k\pi \end{aligned}$$

$$y_{k,n}(N) = B_{k,n}\sin(k\pi nh)$$



Richardson Extrapolation: $\lambda_k = \lambda^e + c_2 h^2 + c_4 h^4 + \ldots$

$$\lambda_k = \lambda_k^{\text{exact}} + ch^2 \qquad \lambda_k(2h) = \lambda_k^e + c4h^2 \qquad \frac{4\lambda_k(h) - \lambda_k(2h)}{3} = \lambda_k^e$$

$$\lambda_1(h=1) = \frac{2}{1}\sin\left(\frac{\pi}{2}\right) = 2$$

$$\lambda_1\left(h=\frac{1}{2}\right) = \frac{2}{(1/2)}\sin\left(\frac{\pi}{4}\right) 2\sqrt{2} = 2.8284271 \qquad \lambda_1\left(h=\frac{1}{4}\right) = 8\sin\left(\frac{\pi}{8}\right) = 3.0614675$$

$$\lambda_k^e \simeq \frac{8\sqrt{2}-2}{3} = 3.10456$$

S	h_2	λ_s^0	λ_s^1	λ_s^2
1	1	2	3.10456	3.1414534
2	1/2	$2\sqrt{2}$	3.1391476	
3	1/4	3.0614675		

$$\lambda_s^0 = \lambda(h_s)$$

$$\lambda_s^{(m)} = \lambda_{s+1}^{(m-1)} + \frac{\lambda_{s+1}^{(m-1)} - \lambda_s^{(m-1)}}{\left(\frac{h_s}{h_{s+m}}\right)^2 - 1}$$

1.2.2. **Numerical solution of ALGEBRAIC EQUATIONS**:

Iterative methods

Consider the solution of

$$Ax = b$$

$$\text{or} \quad \sum_{j=1}^{N} A_{ij} x_j = b_i \qquad i = 1, \ldots, N$$

$$[\ A\ ] = \begin{bmatrix} & 0 \\ L & \end{bmatrix} + \begin{bmatrix} & & 0 \\ & D & \\ 0 & & \end{bmatrix} + \begin{bmatrix} & U \\ 0 & \end{bmatrix}$$

Jacobi Iteration:

$$\sum_{j=1}^{i-1} A_{ij} x_j + A_{ii} x_i + \sum_{j=i+1}^{N} A_{ij} x_j = b_i$$

$$\text{or} \qquad Lx + Dx + Ux = b$$

Iteration Procedure:

$$x_i^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} A_{ij} x_j^{(k)} - \sum_{j=i+1}^{N} A_{ij} x_j^{(k)}\right) \Big/ A_{ii} \Longleftrightarrow \mathbf{x}^{(k+1)} = D^{-1}\left(\mathbf{b} - L\mathbf{x}^{(k)} - u\mathbf{x}^{(k)}\right)$$

$$\text{or} \quad x_i^{(k+1)} = x_i^{(k)} + \left(b_i - \sum_{j=1}^{i-1} A_{ij} x_j^{(k)} - A_{ii} x_i^{(k)} - \sum_{j=i+1}^{N} A_{ij} x_j^{(k)}\right) \Big/ A_{ii} \Leftrightarrow \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + D^{-1}\left(b - Ax^{(k)}\right)$$

$$x^{(k+1)} = x^{(k)} + D^{-1}\left(b - Ax^{(k)}\right)$$

Let $r^{(k)} = b - Ax^{(k)}$ define the residual vector

$= A\left((x^* - x)^{(k)}\right)$

$= Ae^k$ which is a measure of the error.

$\boxed{x^{(k+1)} = x^{(k)} + \omega D^{-1} r^{(k)}}$ where ω is an acceleration parameter.

Jacobi iteration

Eg. 1

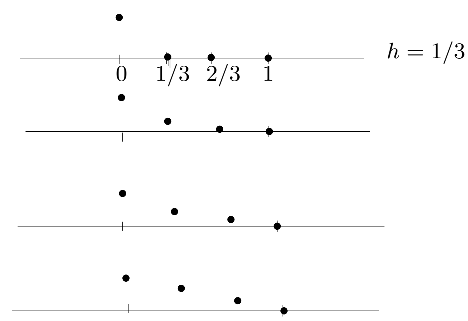
$$\begin{aligned} u'' &= 0 & u_{ex} &= 1-x \\ u(0) &= 1 & u(1) &= 0 \\ \frac{u_{n+1}-2u_n+u_{n-1}}{h^2} &= 0 & u_0 &= 1 \qquad u_N = 0 \end{aligned}$$

$$\begin{array}{ccc} A & u & = \quad b \end{array}$$

$$\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & 0 & \\ & \ddots & & & \\ & 0 & \ddots & & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$$u_n^{(k+1)} = \frac{u_{n+1}^{(k)} + u_{n-1}^{(k)}}{2}$$

Let $u_0^{(0)} = 1 \qquad u_1^{(0)} = 0 \qquad u_2^{(0)} = 0 \qquad u_3^{(0)} = 0 \Leftarrow \text{BC}$



$$\begin{aligned} u_1^{(1)} &= (0+1)/2 = 1/2 \\ u_2^{(1)} &= (0+0)/2 = 0 \end{aligned}$$

$$\begin{aligned} u_1^{(2)} &= (0+1)/2 = 1/2 \\ u_2^{(2)} &= (0+1/2)/2 = 1/4 \end{aligned}$$

$$\begin{aligned} u_1^{(3)} &= (1/4+1)/2 = 5/8 \\ u_2^{(3)} &= (0+1/2)/2 = 1/4 \end{aligned}$$

$$\begin{aligned} u_1^{(4)} &= (1/4+1)/2 = 5/8 = 0.625 \\ u_2^{(4)} &= (0+5/8)/2 = 5/16 = 0.3125 \end{aligned}$$

$$\begin{aligned} u_1^{(5)} &= (5/16+1)/2 = 21/32 = 0.6563 \\ u_2^{(5)} &= (0+5/8)/2 = 5/16 = 0.3125 \\ u_1^{(6)} &= (5/16+1)/2 = 21/32 = 0.6563 \\ u_2^{(6)} &= (0+21/32)/2 = 21/64 = 0.3281. \end{aligned}$$

Gauss Seidel: $\downarrow$ since these are known

$$x_i^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} A_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{N} A_{ij} x_j^{(k)} \right) \Big/ A_{ii}$$

$$\Leftrightarrow \quad \boxed{x^{(k+1)} = D^{-1}\left(b - Lx^{(k+1)} - ux^{(k)}\right)} \text{ or } \boxed{x^{(k+1)} = x^{(k)} + D^{-1}\left(b - Lx^{(k+1)} - Dx^{(k)} - ux^{(k)}\right)}$$

$$\begin{aligned}(D+L)x^{(k+1)} &= Dx^{(k)} + \left(b - Dx^{(k)} - ux^{(k)}\right)\\ &= (D+L)x^{(k)} + \left(b - Lx^{(k)} - Dx^{(k)} - ux^{(k)}\right)\end{aligned}$$

$$\therefore \quad x^{(k+1)} = x^{(k)} + (D+L)^{-1}\left(b - Ax^{(k)}\right)$$

$$\boxed{x^{(k+1)} = x^{(k)} + (D+L)^{-1} r^{(k)}} \Leftarrow \text{ Interpretation.}$$

Successive-over-Relaxation (SOR):

$$x^{(k+1)} = x^{(k)} + \omega D^{-1}\left(b - Lx^{(k+1)} - Dx^{(k)} - Ux^{(k)}\right).$$

ω acceleration parameter
$\omega = 1 \Rightarrow GS.$

Interpretation:

$$(\omega^{-1}D + L)x^{(k+1)} = (\omega^{-1}D + L)x^{(k)} + \left(b - Lx^{(k)} - Dx^{(k)} - ux^{(k)}\right)$$

$$\therefore x^{(k+1)} = x^{(k)} + \left(\omega^{-1}D + L\right)^{-1}(b - Ax^k)$$

$$\boxed{x^{(k+1)} = x^{(k)} + \left(\omega^{-1}D + L\right)^{-1} r^{(k)}.}$$

Gauss Seidel Iteration

Eg:

$$u'' = 0 \qquad u = 1 - x$$

$$u(0) = 1\ ; \qquad u(1) = 0$$

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = 0$$

$$\begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ 0 & 1 & -2 & 1 & & & \\ & & & & & & \\ & & & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ \\ \vdots \\ \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \\ \vdots \\ 0 \end{bmatrix}$$

$$Au = b$$

$$u_n^{(k+1)} = \frac{\left(u_{n+1}^{(k)} + u_{n-1}^{(k+1)}\right)}{2}$$

Let

$$
\begin{aligned}
u_0^{(0)} &= 1 \qquad u_1^{(0)} = 0 \qquad u_2^{(0)} = 0 \qquad u_3^{(0)} = 0 \Leftarrow BC \\
u_1^{(1)} &= (0+1)/2 = 1/2 = 0.5 \\
u_2^{(1)} &= (0+1/2)/2 = 1/4 = 0.25
\end{aligned}
$$

FIGURE

$$
\begin{aligned}
u_1^{(1)} &= \frac{u_2^{(0)} + u_0^{(1)}}{2} \\
u_2^{(1)} &= \frac{u^{(0)} + u_1^{(1)}}{2}
\end{aligned}
$$

$$
\begin{aligned}
u_1^{(2)} &= (1+1/4)/2 = 5/8 = 0.625 \\
u_2^{(2)} &= (0+5/8)/2 = 5/16 = 0.3125
\end{aligned}
$$

$$
\begin{aligned}
u_1^{(3)} &= (1+5/16)/2 = 21/32 = 0.6563 \\
u_2^{(3)} &= (0+21/32)/2 = 21/64 = 0.3281
\end{aligned}
$$

$$
\begin{aligned}
u_1^{(4)} &= (1+21/64)/2 = 85/128 = 0.6641 \\
u_2^{(4)} &= (0+85/128)/2 = 85/256 = 0.3320
\end{aligned}
$$

$$
\begin{aligned}
u_1^{(5)} &= (1+85/256)/2 = 341/512 = 0.6660 \\
u_2^{(5)} &= (0+341/512)/2 = 341/1024 = 0.3330.
\end{aligned}
$$

General iterative method:

$$
x^{(k+1)} = x^{(k)} + \alpha_k B^{-1} r^{(k)} \text{ where } r^k = b - Ax^k.
$$

$$
\begin{aligned}
\alpha_k &\equiv 1 \qquad B^{-1} = D^{-1} \Rightarrow \text{ Jacobi} \\
\alpha_k &= 1 \qquad B^{-1} = (\omega^{-1} D + L) \Rightarrow \text{ SOR and Gauss Seidel.} \\
\alpha_k &= 1 \qquad B^{-1} = A^{-1} \Rightarrow \text{ Newton's method (vacuous in this case).} \\
r^{(k+1)} &= b - Ax^{k+1} \\
&= b - A\left(x^{(k)} + \alpha_k B^{-1} r^{(k)}\right) \\
&= r^{(k)} - \alpha_k A B^{-1} r^{(k)} \\
&= \left(I - \alpha_k A B^{-1}\right) r^{(k)} \\
&= \left(I - \alpha_k A B^{-1}\right)\left(I - \alpha_{k-1} A B^{-1}\right) r^{(k-1)}
\end{aligned}
$$

$$
\boxed{r^{(k+1)} = \prod_{s=1}^{k} \left(I - \alpha_s A B^{-1}\right) r^{(1)} = P_k(AB^{-1}) r^{(1)}}
$$

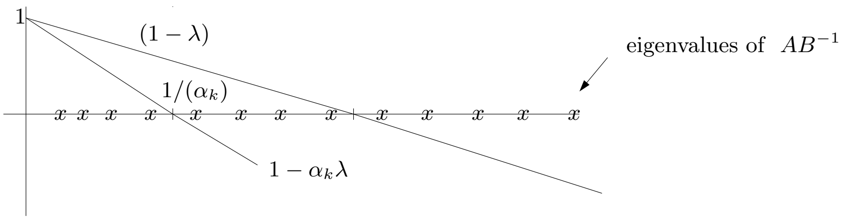
where $P_k(\hat{A}) = \prod_{s=1}^{k} \left(I - \alpha_s \hat{A}\right)$ is a polynomial of degree k in $\hat{A}$.

Let $\{\lambda_j\}$ be the eigenvalues and $\{v_j\}$ be the corresponding eigenvectors of $\hat{A} = AB^{-1}$: i.e. $\hat{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j$. Then expanding $\mathbf{r}^1$ and $\mathbf{r}^{(k+1)}$ in terms of $\{v_j\}$:

$$
\mathbf{r}^{(1)} = \sum_{j=1}^{N} \hat{r}_j^{(1)} \mathbf{v}_j \quad \text{and} \quad \mathbf{r}^{(k+1)} \sum_{j=1}^{N} \hat{r}^{(k+1)} \mathbf{v}_j
$$

we obtain:

$$\hat{r}_j^{(k+1)} = \prod_{s=1}^{k} (1 - \alpha_s \lambda_j)\, \hat{r}_j^{(1)} = P_k(\lambda_j)\hat{r}_j^{(1)}$$



Note: For Jacobi and SOR $\quad \alpha_k \equiv 1$ so that $P_k(\lambda) = (1-\lambda)^k$

$$\begin{aligned}
|r^{(k+1)}|^2 &= \left| \sum_i (1-\lambda_j)\, \hat{r}_j^{(k)} v_j \right|^2 \\
&\le |1-\hat{\lambda}|^2 \left| \sum_j \hat{r}_j^{(k)} v_j \right|^2 \qquad \hat{\lambda} : |1-\hat{\lambda}| = \max\{|1-\lambda_1|,\, |1-\lambda_N|\}
\end{aligned}$$

$$\boxed{|r^{k+1}| \le \rho |r^{(k)}|} \text{ where } \rho = \max\{|1-\lambda_1|,\, |1-\lambda_N|\}.$$

Example of degredation of Jacobi with mesh refinement.

$$\begin{aligned}
-u'' &= f \\
A \cdot u_n &= -u_{n-1} + 2u_n - u_{n-1} = h^2 f_n \\
\lambda_k &= 4\sin^2\left(\frac{k\pi}{2N}\right) \qquad \text{are the eigenvalues of } A \\
AD^{-1} &= \frac{-E^{-1} + 2 - E}{2} \Rightarrow \mu_1 = 2\sin^2\left(\frac{\pi}{2N}\right) \overset{N \gg 1}{\approx} \frac{\pi^2}{2N^2} \\
\therefore \qquad & \rho \overset{N \gg 1}{\approx} 1 - \frac{\pi^2}{2N^2}.
\end{aligned}$$

We can expect poor performance as N increases. Look for the number of iterations it will take to achieve a tolerance ε:

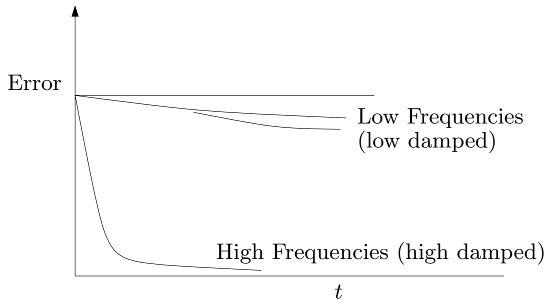
$$\begin{aligned}
\rho^r &= \varepsilon \\
r &= \frac{\ln \varepsilon}{\ln \rho} = \frac{\ln \varepsilon}{\ln\left(1 - \frac{\pi^2}{2N^2}\right)} = \frac{\ln \varepsilon}{-\frac{\pi^2}{2N^2}\left(1 + \frac{1}{2}\left(\frac{\pi^2}{2N^2}\right) + \dots\right)} \sim -\frac{2N^2}{\pi^2} \ln \varepsilon
\end{aligned}$$

Physical interpretation of Jacobi's method as a diffusion process:

$$\begin{aligned}
u_n^{(k+1)} &= \frac{u_{n+1}^{(k)} + u_{n-1}^{(k)}}{2} \\
\therefore u_n^{(k+1)} - u_n^{(k)} &= \frac{u_{n+1}^{(k)} - 2u_n^{(k)} + u_{n-1}^{(k)}}{2} \\
\therefore \frac{u_n^{(k+1)} - u_n^{(k)}}{\Delta t} &= \left(\frac{h^2}{2\Delta t}\right) \frac{u_{n+1}^{(k)} - 2u_n^{(k)} + u_{n-1}^{(k)}}{h^2} \overset{h,\Delta t \Rightarrow 0}{\longleftrightarrow} \boxed{\frac{\partial u}{\partial t} = \frac{D\partial^2 u}{\partial x^2}}
\end{aligned}$$

Fourier analysis:

$$\begin{aligned}\frac{\partial \hat{u}}{\partial t} &= -D\omega^2 \hat{u} \\ \hat{u} &= \hat{u}_0 e^{-D\omega^2 t}\end{aligned}$$



Minimization approach to solving linear equations:

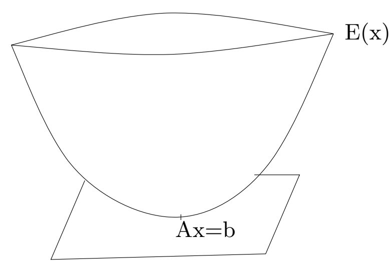
Instead of solving $Ax = b$, consider the equivalent problem of minimzing the quadratic form

$$E(x) = \frac{1}{2}x^T Ax - x^T b.$$

For a minimum we have the necessary conditions

$$0 = \frac{\partial E}{\partial x} = Ax - b.$$

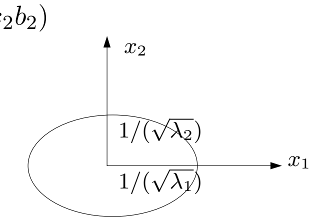
Let A be symmetric and positive definite, then the eigenvalues λ_k of A are all real and positive. So $E(x)$ can be viewed as a parabolic surface with elliptic cross sections.



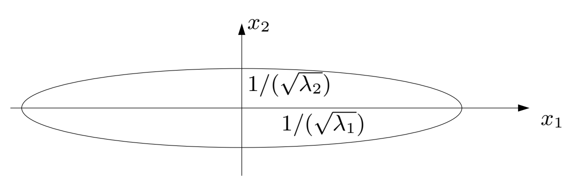
2D Example:

$$\begin{aligned}A &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ E &= \frac{1}{2}x^T Ax - x^T b = \frac{1}{2}\left(\lambda_1 x_1^2 + \lambda_2 x_2^2\right) - (x_1 b_1 + x_2 b_2)\end{aligned}$$

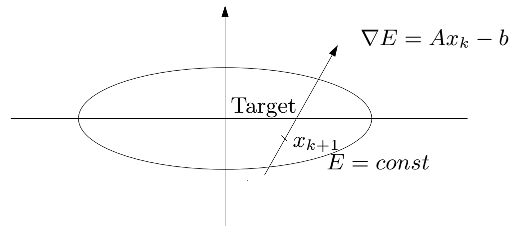
Level sets of E are ellipses



What happens if $\lambda_2 \gg \lambda_1$



Steepest descent algorithm:



Idea: Search for a minimum along the path defined by $\nabla E = Ax - b$

Consider the so-called Richardson Scheme:

$$x_{k+1} = x_k + \alpha_k(b - Ax_k). \qquad \text{We must look in the steepest descent direction} - \nabla E.$$

$$\begin{aligned} \text{Choose} \quad & \alpha_k \quad \text{to minimize } E: \\ E(x_{k+1}) &= x_{k+1}^T A x_{k+1} \\ &= (x_k + \alpha r_k)^T A (x_k + \alpha r_k) \\ 0 = \frac{\partial E}{\partial \alpha} &= 2 r_k^T A (x_k + \alpha r_k) \Rightarrow \quad \alpha_k = -\frac{r_k^T A x_k}{r_k^T A r_k}. \end{aligned}$$

Algorithm: Steepest descents.

$$x_{k+1} = x_k + \alpha_k r_k \quad \text{where} \quad \alpha_k = -\frac{r_k^T A x_k}{r_k^T A r_k}.$$

Notice:

- The similarity to the general iterative method, in this case $B = I$.
- The role of the preconditioner is to try to make all the eigenvalues of AB^{-1} as close as possible to 1. In this case the ellipses $\sim$ circles and the steepest descent method will converge rapidly.