**Math 405/607E Fall 2009**
**Assignment 3: Solving initial value ODE**
**The due date for this assignment is Friday November 13 th.**

1. Use the MATLAB routines `euler.m` (FE) and `BackEuler.m` (BE) posted on the web site as templates for explicit and implicit solvers in order to develop your own versions of routines for the following methods: Improved Euler (ImpE), the Trapezium Rule (TR), and RK-4 that can be used to integrate the initial value ODE:

$$y' = f(x, y), \quad y(a) = y_0 \tag{1}$$

on an interval $[a, b]$.

(a) Use FE and ImpE to solve: $[a, b] = [0, \frac{1}{2}]$, $y' = 10(-1-3^{\frac{1}{2}}i)y$, $y(0) = 1$, $y_{\text{exact}}(x) = e^{10(-1-\sqrt{3}i)x}$. Use the stability region in each case determine the maximal value of the step-size $h_m$ for which each method will be stable. Solve the problem with $h < h_m$, $h = h_m$, $h > h_m$ to test the theoretical stability results.

(b) Consider the initial value problem (IVP): $[a, b] = [0, 1]$, $y' = \frac{x}{1+x^2}y$, $y(0) = 1$, $y_{\text{exact}}(x) = (1 + x^2)^{1/2}$. A useful way to determine the convergence rate of a given method, without plotting, is to determine $Y_N(h)$ and $Y_{2N}(h/2)$ using the same method and evaluating following quotient

$$Q = \log_2 \frac{E(h)}{E(h/2)} \tag{2}$$

where $E(h) = |y_{\text{exact}}(b) - Y_N(h)|$. Assuming that $E(h) = ch^p$ show that the quotient $Q$ given in (2) is just $p$ - the order of the method. Now complete the following table for the following methods by solving the above ODE to obtain estimates of $y_{\text{exact}}(1) = 2^{1/2}$.

| $Method$ | $h = 2^{-2}$ | $h = 2^{-3}$ | $h = 2^{-4}$ |
|---|---|---|---|
| $E_{FE}$ | | | |
| | $Q_{FE}$ | | |
| $E_{\text{ImpE}}$ | | | |
| | $Q_{\text{ImpE}}$ | | |
| $E_{\text{TR}}$ | | | |
| | $Q_{\text{TR}}$ | | |
| $E_{\text{RK4}}$ | | | |
| | $Q_{\text{RK4}}$ | | |
| $E_{\text{BE}}$ | | | |
| | $Q_{\text{BE}}$ | | |

2. Consider the difference scheme

$$Y_{k+2} = Y_k + \frac{h}{2}(f_{k+1} + 3f_k) \tag{3}$$

Determine:

(a) The truncation error for (3).

(b) The stability region for (3) by considering the model problem in which: $f(x, y) = \lambda y,\ y(0) = 1$ and converting the routine `StabregionsImplicit` posted on the web.

(c) Make the substitution $Y_k = G^k$ in the scheme applied to the model problem and parameterize the boundary of the stability region $|G| = 1$ by making the substitution $G = e^{i\theta}$. Now obtain an expression for the so-called boundary locus $z = h\lambda = g(\theta)$. How does this compare with the stability region obtained in (b)?

3. Consider the following model problem for a stiff ODE:

$$y' = -\alpha(y - \sin(x)) + \cos(x), \quad y(0) = 1$$

for which the exact solution is $y = \sin x + e^{-\alpha x}$. Observe that there are two very different time scales built into this problem.

(a) Use your RK2/ImpE code, the Crank-Nicolson method, and the Backward Euler scheme to solve this problem with $\alpha = 1000$ on the interval $[0, \pi]$. Start with about 20 steps for each of these methods. Plot each of these solutions and compare them to the exact solution. Explain each of the phenomena you observe.

(b) Determine theoretically how many steps will be required for the RK2 algorithm to work. Confirm these results with a numerical experiment.

(c) Will adaptive time stepping alleviate the problem? Use the MATLAB routine ode23 to solve this problem to a tolerance of $10^{-4}$. How many steps does it require? What can you conclude from this example?

4. Consider the following parametric representation for a circle of radius $r$: $(x, y) = (r\cos\theta, r\sin\theta)$. Differentiating with respect to $\theta$ we obtain the following system of ODE:

$$\begin{aligned} x' &= -y,\ x(0) = r \\ y' &= x,\ y(0) = 0 \end{aligned}$$

whose solution trajectory $(x(\theta), y(\theta))$ is, by construction, a circle of radius $r$.

(a) Use the Forward Euler, the Backward Euler, and the trapezoidal methods to solve this system and plot the solution trajectories $(x(\theta), y(\theta))$. In your simulations use $h = 0.02$ over the interval $0 \leq \theta \leq 120$. Is the expected circle reproduced in each case?

(b) By multiplying the first equation by $x$ and the second equation by $y$, adding the two resulting equations, and integrating, show that the quantity $r^2 =$ constant is a conserved quantity of the system of ODE (given the source of the ODE this is to be expected). By performing a similar analysis on each of the difference equations for the numerical schemes in (a), determine the extent to which they will conserve the radius. Use the results of your analysis to explain the numerical results that you obtained in (a).