# Math 405/607 E: Ass. 1: Due 7 Oct 2009

1. **Finite Difference Tables:** Let $S_N^k$ denote the sum of the $k$ th powers of the first $N$ integers i.e.:

$$S_N^k = \sum_{i=1}^{N} i^k$$

Write a simple MATLAB program to evaluate these sums for a specified value of $k$ for values of $N$ from $1\ldots k+3$. Now write MATLAB code to form the forward difference table (since the sample points are uniform). Notice that for each value of $k$ the difference table terminates - why does this happen? For the special case $k = 4$ extract the differences from your table and use the Gregory-Newton divided difference formula to derive the formula:

$$S_N^4 = \sum_{i=1}^{N} i^4 = \frac{1}{30}N(2N+1)(N+1)(3N^2+3N-1)$$

**SOLUTION:**

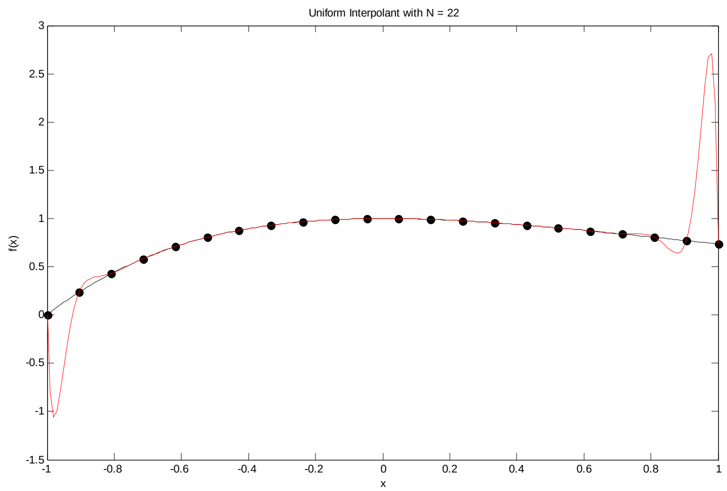
| $N$ | $S_N^k$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ | $\Delta^4 f$ | $\Delta^5 f$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 15 | 50 | 60 | 24 |
| 1 | 1 | 16 | 65 | 110 | 84 | 24 |
| 2 | 17 | 81 | 175 | 194 | 108 | 24 |
| 3 | 354 | 625 | 369 | 302 | 132 | |
| 4 | 979 | 1296 | 1105 | | | |
| 5 | 2275 | 2401 | | | | |
| 6 | 4676 | | | | | |

$$S_N^4 = N + \frac{N(N-1)}{2}15 + \frac{N(N-1)(N-2)}{3!}50 + \frac{N(N-1)(N-2)(N-3)}{4!}60 + \frac{N(N-1)(N-2)(N-3)(N-4)}{5!}24$$
$$= \frac{1}{30}N(2N+1)(N+1)(3N^2+3N-1)$$

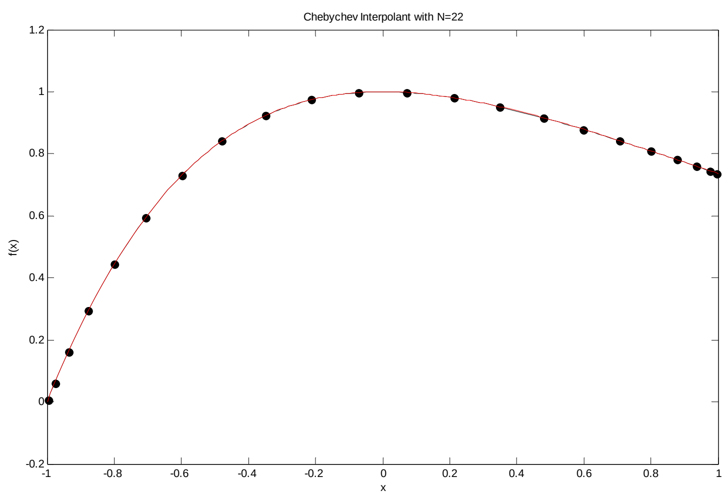
2. **Sensitivity of Polynomial Interpolation to perturbations:** Use the demo code posted on the web site and write a routine yi=Nddiff(x,y,xi) to determine the Newton divided difference polynomial interpolant of a function y=f(x) whose values at the vector of sample points x are given in the vector y, while xi is the vector at which the desired interpolated values are requested. Illustrate your results with the function $f(x) = (1+x)\exp(-x)$ on the interval $[-1, 1]$ with $N = 22$ uniformly distributed points. Now add a small random perturbation to each of the sampled values using y+10^(-3)*rand(1,N). Plot the interpolant of the perturbed function. If at first the interpolant seems fine, repeat the run since by chance the random perturbations may not have been significant. What happens if you increase/decrease $N$? What happens if you

sample the function at the zeros of the Chebychev polynomial of degree 21 instead of the uniformly distributed points?

SOLUTION:




If we increase the number of sample points for the uniform mesh the oscillations at the endpoints due to the perturbations increase dramatically.

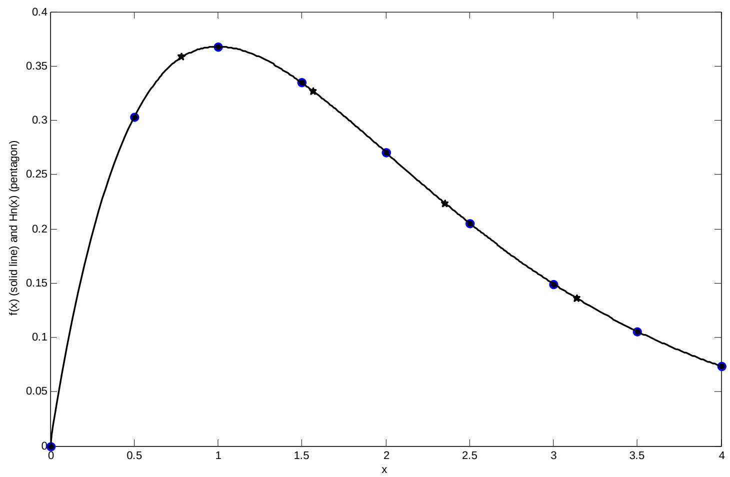


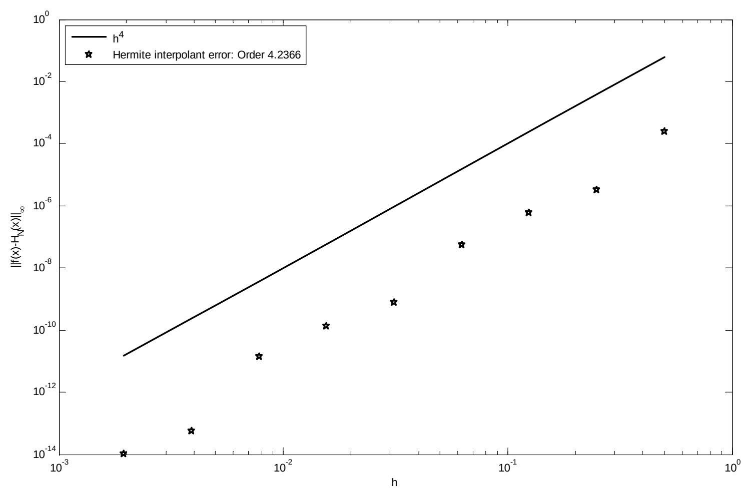

3. **Hermite Cubic Interpolation:** Write a MATLAB function fi=fhermite(x,f,fp,xi) to evaluate the piecewise cubic Hermite interpolant of the function whose values and first derivatives at the sample points x are stored in the vectors f and fp respectively. The function should evaluate the piecewise Hermite polynomials at the specified points xi and store the result in the vector fi. As a test use the routine to interpolate the function $f(x) = xe^{-x}$ on the interval $[0,4]$. Repeat the interpolation with mesh spacing $h = 2.\hat{}(-9 : 1 : -1)$ and provide a log-log plot of the infinity norm of the error (i.e. $\|f - H_N\|_\infty$) measured at the interpolated points xi $=[\pi/4, \pi/2, 3\pi/4, \pi]$ against the mesh size $h$. What is the rate of convergence of piecewise cubic Hermite interpolation? (Hint:it may be useful to use the find($\bullet$) function in MATLAB). Complete the following table of values of the Hermite cubic interpolant $H_N(x)$ obtained by sampling the function $f(x)$ at

x=0:0.25:4.

| $x$ | $\pi/4$ | $\pi/2$ | $3\pi/4$ |
|---|---|---|---|
| $H_N(x)$ | | | |

SOLUTION:




The rate of convergence is $O(h^4)$. Using sample points take nat x=0:0.25:4 the follwing interpolated values are obtained.

| $x$ | $\pi/4$ | $\pi/2$ | $3\pi/4$ |
|---|---|---|---|
| $f(x)$ | 0.3581 | 0.3265 | 0.2233 |
| $H_N(x)$ | 0.3627 | 0.3365 | 0.2278 |

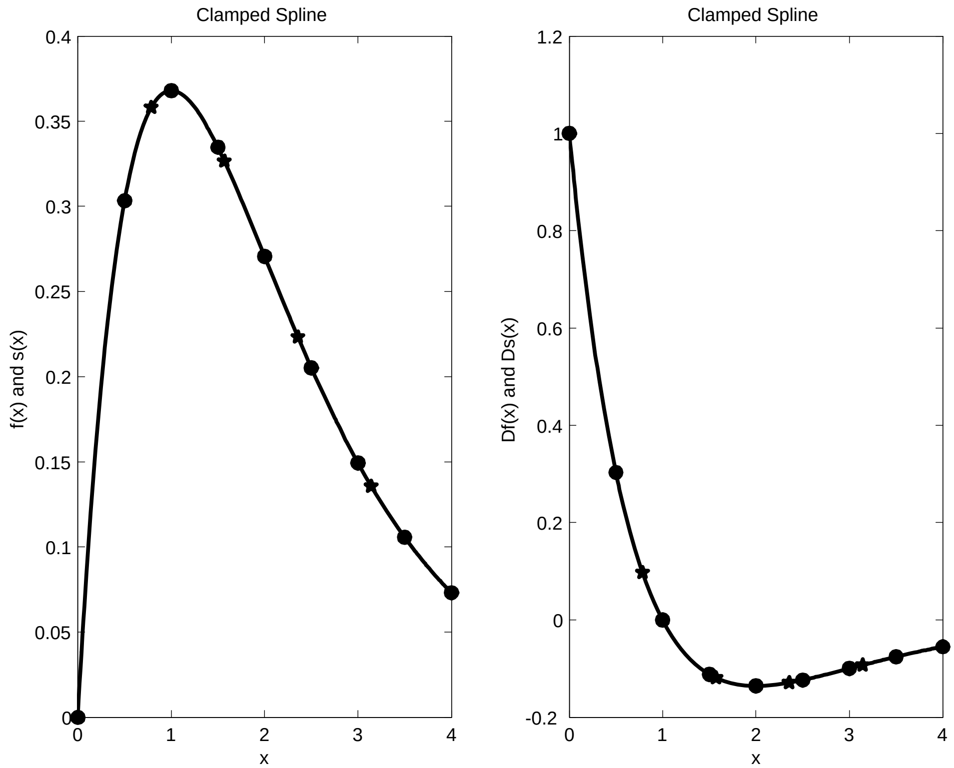
**4. Spline Interpolation:** Write a MATLAB function [s,sp]=fcspline(x,f,fpa,fpb,xi,itype) to evaluate the **clamped** cubic spline interpolant of the function whose values at the sample points x are stored in the vectors f, and whose derivatives at the endpoints of the interval fpa=f(x(1)) and fpb=f(x(end)) are given as input. Also write a MATLAB function [s,sp]=fnakspline(x,f,xi) to evaluate the **not-a-knot** cubic spline interpolant of the function whose values at the sample points x are stored in the vectors f. The functions should evaluate the spline interpolants and their first derivatives at the specified points xi and store the results in the vectors s and sp respectively. As a test case use the routine to interpolate the function $f(x) = xe^{-x}$ on the interval $[0,4]$. Plot the interpolants of $f$ and $f'$ for the case $h = 1/2$ using both splines. Now for both splines repeat the interpolation with $h = 2.\hat{}(-6 : 1 : -4)$ and provide a log-log plot of the infinity norms of the errors (i.e. $\|f(x) - s_N(x)\|_\infty$ measured at the interpolated points xi $=[\pi/4, \pi/2, 3\pi/4, \pi]$ and $\|f'(x) - s'_N(x)\|_\infty$ measured at the interpolated points xi and at the sample points x) against the mesh size $h$. Tabulate your results as follows:

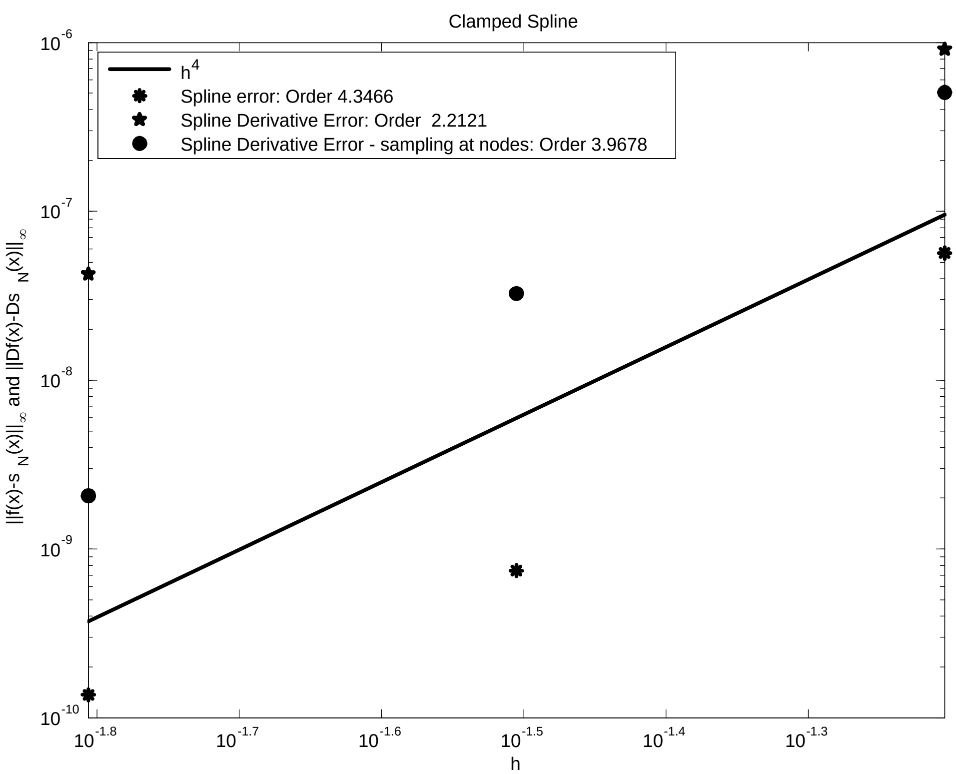
| $h$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ |
|---|---|---|---|
| $\|f(\text{xi}) - s_N(\text{xi})\|_\infty$ | | | |
| $\|f'(\text{xi}) - s'_N(\text{xi})\|_\infty$ | | | |
| $\|f'(\text{x}) - s'_N(\text{x})\|_\infty$ at sample points x | | | |

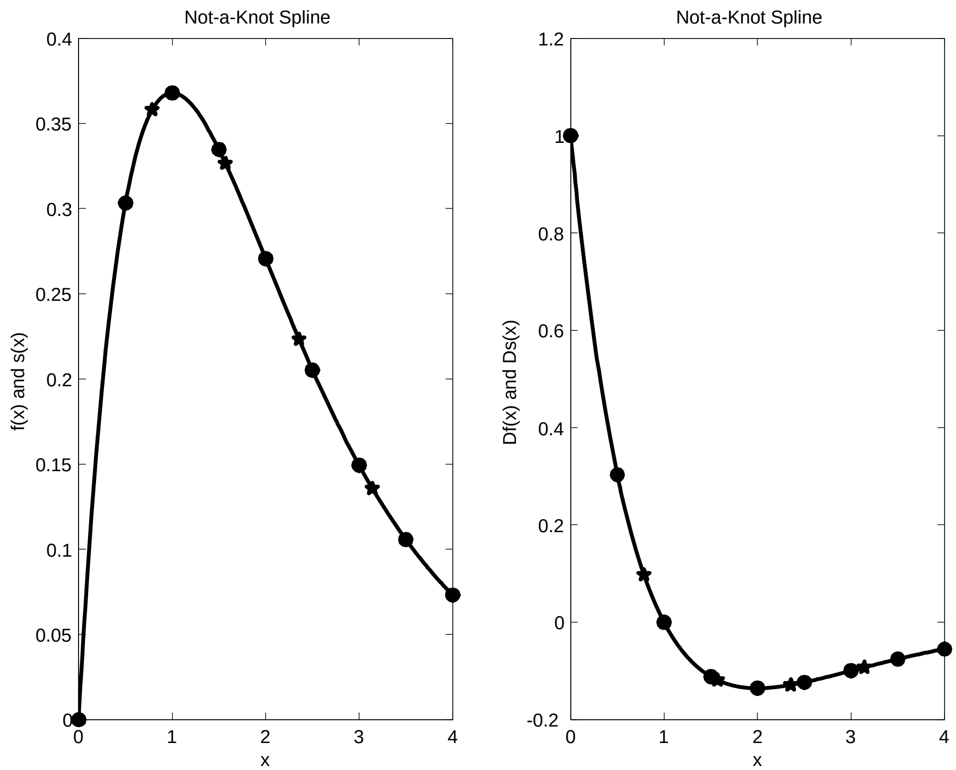
What is the rate of convergence of these piecewise cubic spline interpolation and derivative approximation? Does this agree with the theory?

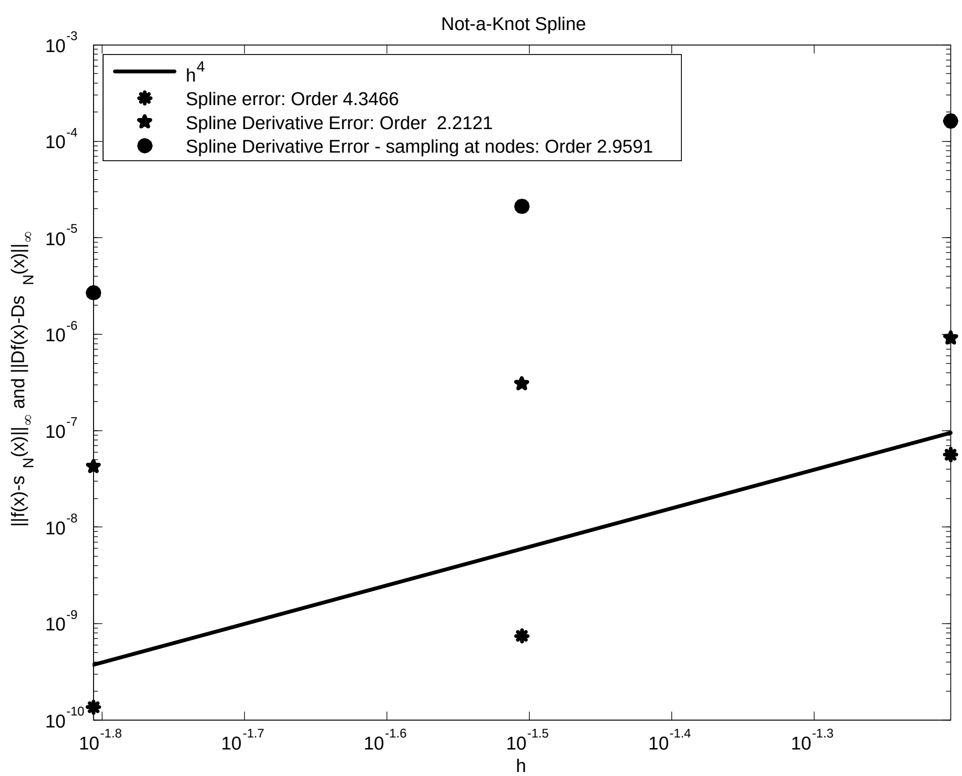
SOLUTION:

The interpolated values are denoted by (*) in the figure below while the sample points are denoted by the solid circles.


Clamped Spline
f(x) and s(x)
x
Clamped Spline
Df(x) and Ds(x)
x


Clamped Spline
h^4
Spline error: Order 4.3466
Spline Derivative Error: Order 2.2121
Spline Derivative Error - sampling at nodes: Order 3.9678
||f(x)-s_N(x)||_∞ and ||Df(x)-Ds_N(x)||_∞
h




For the clamped spline:

| $h$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | ERROR Order |
|---|---|---|---|---|
| $\|f(\text{xi}) - s_N(\text{xi})\|_\infty$ | $5.67\text{x}10^{-8}$ | $7.46\text{x}10^{-10}$ | $1.36\text{x}10^{-10}$ | 4.35 |
| $\|f'(\text{xi}) - s_N'(\text{xi})\|_\infty$ | $9.13\text{x}10^{-7}$ | $3.08\text{x}10^{-7}$ | $4.25\text{x}10^{-8}$ | 2.21→ 3 |
| $\|f'(\text{x}) - s_N'(\text{x})\|_\infty$ at sample points x | $5.07\text{x}10^{-7}$ | $3.26\text{x}10^{-8}$ | $2.07\text{x}10^{-9}$ | 3.96 |

For the not-a-knot spline:

| $h$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | ERROR Order |
|---|---|---|---|---|
| $\|f(\text{xi}) - s_N(\text{xi})\|_\infty$ | $5.67\text{x}10^{-8}$ | $7.46\text{x}10^{-10}$ | $1.3759\text{x}10^{-10}$ | 4.35 |
| $\|f'(\text{xi}) - s_N'(\text{xi})\|_\infty$ | $9.13\text{x}10^{-7}$ | $3.08\text{x}10^{-7}$ | $4.25\text{x}10^{-8}$ | 2.21 → 3 |
| $\|f'(\text{x}) - s_N'(\text{x})\|_\infty$ at sample points x | $1.63\text{x}10^{-4}$ | $2.11\text{x}10^{-5}$ | $2.69\text{x}10^{-6}$ | 2.96 |

The function values for both splines are $O(h^4)$ and the derivatives are measured at the few sample points xi $O(h^{2.21})$ if more sample points are taken this becomes. $O(h^3)$. At the sample points x the clamples spline achieves an $O(h^4)$ in the derivatives too (because exact boundary conditions are given). However, the not-a-knot spline only manages an $O(h^3)$ estimate of $f'$ because oof the error in the two additional boundary conditionsin the not-a-knot approximation.

5. **Chebychev Points:** The so-called Chebychev points

$$x_j = \cos(j\pi/N), \qquad j = 0, 1, \ldots, N,$$

which are the points at which the Chebychev polynomials take on extreme values in $[-1, 1]$ (check this), are commonly used in spectral approximations for non-periodic functions. Derive the density function $\rho_N(x)$ associated with these points? Using this function can you predict the performance of polynomial approximation sampled at these points. Use your Nddiff routine developed in Q2 to interpolate the function $f(x) = (1-x)^{1/3}$ at the points xi=-1:0.005:1 by sampling the function at the Chebychev points for $N = 17$, and at the Chebychev roots for $N = 17$. Plot the interpolants and the errors at the xi. How do these interpolants and their errors compare? Now repeat this experiment with $g(x) = (4-x)^{1/3}$. Explain the difference between the interpolatns for $f$ and $g$.

SOLUTION: Consider two distinct Chebychev points $x_k$ and $x_q$ :

$$x_k = \cos(k\pi/N) \text{ and } x_q = \cos(q\pi/N) = x_k + \Delta x$$

Then inverting and taking the difference we obtain:

$$\begin{aligned} k - q &= \frac{N}{\pi}(\cos^{-1}x_k - \cos^{-1}(x_k + \Delta x)) \\ &\approx \frac{N\Delta x}{\pi(1-x_k^2)^{\frac{1}{2}}} \end{aligned}$$

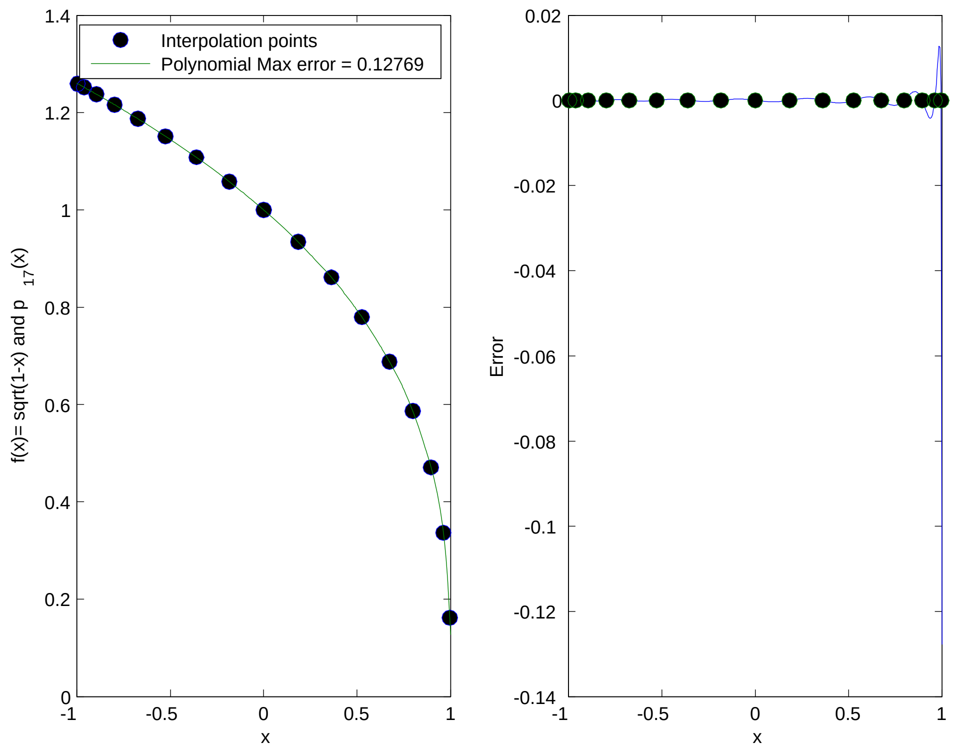
Thus the density per unit length is given by:

$$\rho_N(x) = \frac{k-q}{\Delta x} = \frac{N}{\pi(1-x^2)^{\frac{1}{2}}}$$

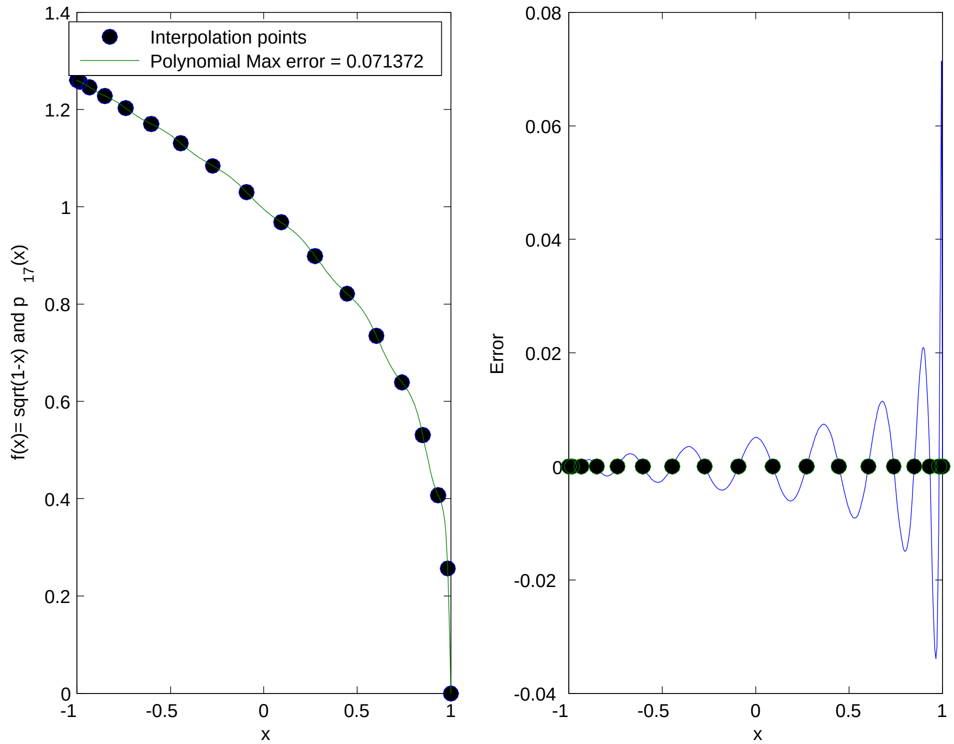
Since the density function is the same as that for the Chebychev roots, we can expect the same interpolation performance from the Chebychev points.

Because the derivatives of $f$ are singular at $x = 1$ the approximation close to this point is poor and the approximation to $f$ is much worse than that for $g$, which is analytic at $x = 1$. (see below)

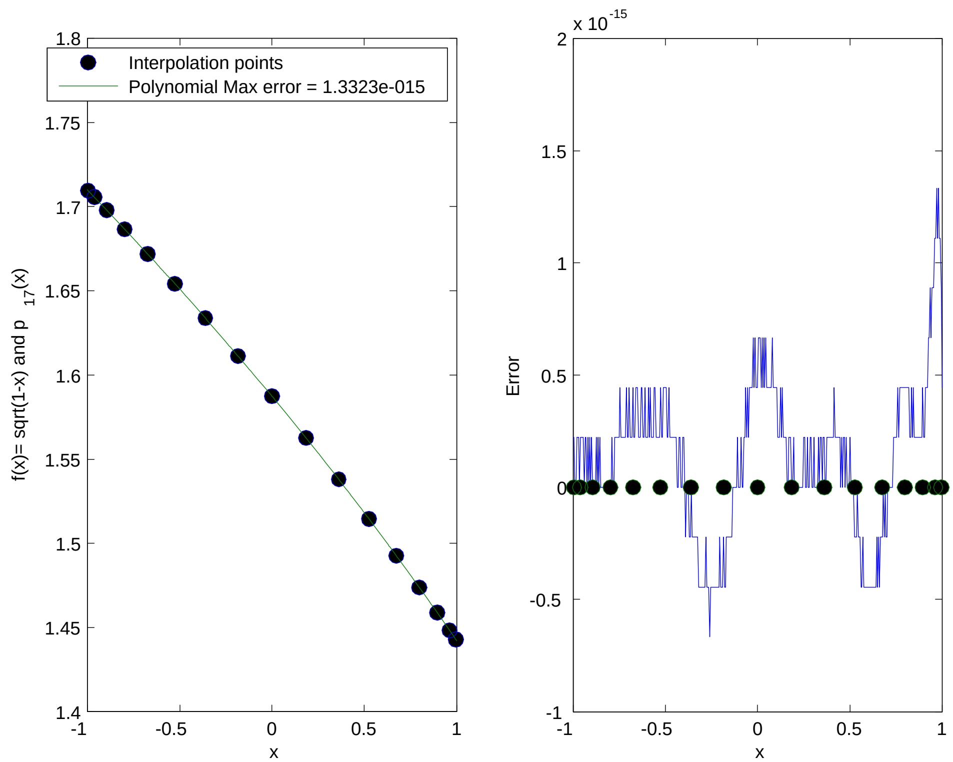
CHEBYSHEV ROOTS: Approximation to $f$:



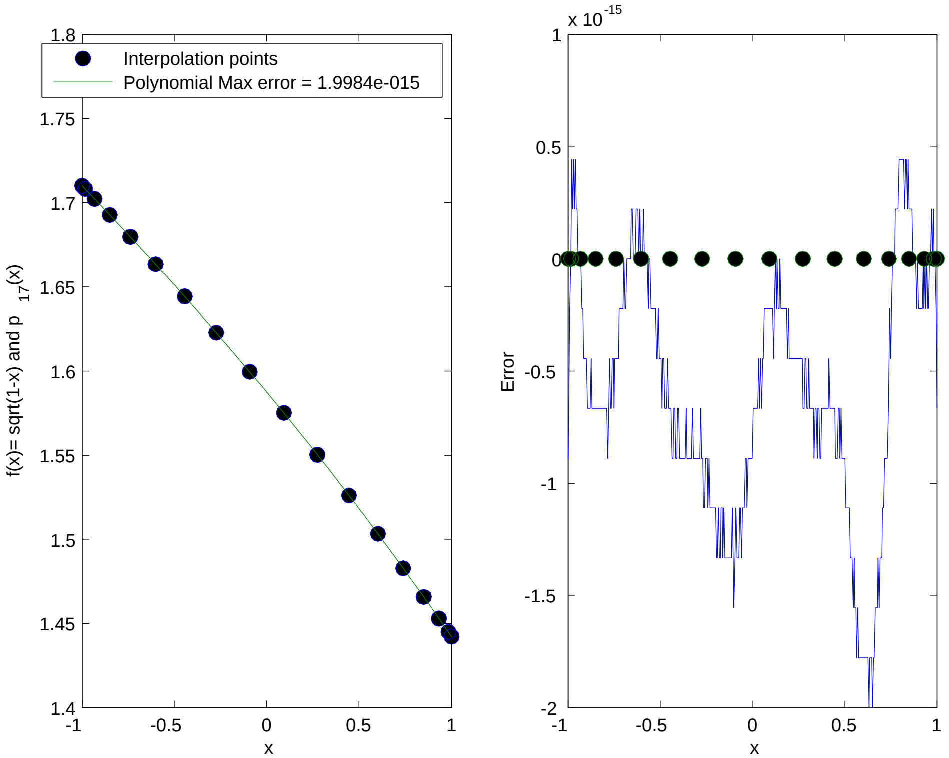

CHEBYSHEV POINTS: Approximation to $f$:




CHEBYSHEV ROOTS: Approximation to *g*




CHEBYSHEV POINTS: Approximation to *g*




For the smooth function $g$ the Chebychev points and Chebychev roots both achieve approximations with errors clsoe to roundoff because they share the same asymptotic density function $\rho_N(x)$..