1 Math 405/607 E: Ass. 1: Due 7 Oct 2009

1. **Finite Difference Tables:** Let S_N^k denote the sum of the k th powers of the first N integers i.e.:

$$S_N^k = \sum_{i=1}^{N} i^k$$

Write a simple MATLAB program to evaluate these sums for a specified value of k for values of N from $1 \ldots k+3$. Now write MATLAB code to form the forward difference table (since the sample points are uniform). Notice that for each value of k the difference table terminates - why does this happen? For the special case $k = 4$ extract the differences from your table and use the Gregory-Newton divided difference formula to derive the formula:

$$S_N^4 = \sum_{i=1}^{N} i^4 = \frac{1}{30} N\left(2N+1\right)\left(N+1\right)\left(3N^2+3N-1\right)$$

2. **Sensitivity of Polynomial Interpolation to perturbations:** Use the demo code posted on the web site and write a routine yi=Nddiff(x,y,xi) to determine the Newton divided difference polynomial interpolant of a function y=f(x) whose values at the vector of sample points x are given in the vector y, while xi is the vector at which the desired interpolated values are requested. Illustrate your results with the function $f(x) = (1+x)\exp(-x)$ on the interval $[-1, 1]$ with $N = 22$ uniformly distributed points. Now add a small random perturbation to each of the sampled values using y+10^(-3)*rand(1,N). Plot the interpolant of the perturbed function. If at first the interpolant seems fine, repeat the run since by chance the random perturbations may not have been significant. What happens if you increase/decrease N? What happens if you sample the function at the zeros of the Chebychev polynomial of degree 21 instead of the uniformly distributed points?

3. **Hermite Cubic Interpolation:** Write a MATLAB function fi=fhermite(x,f,fp,xi) to evaluate the piecewise cubic Hermite interpolant of the function whose values and first derivatives at the sample points x are stored in the vectors f and fp respectively. The function should evaluate the piecewise Hermite polynomials at the specified points xi and store the result in the vector fi. As a test use the routine to interpolate the function $f(x) = xe^{-x}$ on the interval $[0, 4]$. Repeat the interpolation with mesh spacing $h = 2.\hat{}(-9 : 1 : -1)$ and provide a log-log plot of the infinity norm of the error (i.e. $||f - H_N||_\infty$) measured at the interpolated points xi $=[\pi/4, \pi/2, 3\pi/4, \pi]$ against the mesh size h. What is the rate of convergence of piecewise cubic Hermite interpolation? (Hint:it may be useful to use the find($\cdot$) function

in MATLAB). Complete the following table of values of the Hermite cubic interpolant $H_N(x)$ obtained by sampling the function $f(x)$ at x=0:0.25:4.

x	$\pi/4$	$\pi/2$	$3\pi/4$
$H_N(x)$			

4. **Spline Interpolation:** Write a MATLAB function [s,sp]=fcspline(x,f,fpa,fpb,xi,itype) to evaluate the **clamped** cubic spline interpolant of the function whose values at the sample points x are stored in the vectors f, and whose derivatives at the endpoints of the interval fpa=f(x(1)) and fpb=f(x(end)) are given as input. Also write a MATLAB function [s,sp]=fnakspline(x,f,xi) to evaluate the **not-a-knot** cubic spline interpolant of the function whose values at the sample points x are stored in the vectors f. The functions should evaluate the spline interpolants and their first derivatives at the specified points xi and store the results in the vectors s and sp respectively. As a test case use the routine to interpolate the function $f(x) = xe^{-x}$ on the interval $[0, 4]$. Plot the interpolants of f and f' for the case $h = 1/2$ using both splines. Now for both splines repeat the interpolation with $h = 2.\hat{}(-6 : 1 : -4)$ and provide a log-log plot of the infinity norms of the errors (i.e. $||f(x) - s_N(x)||_\infty$ measured at the interpolated points xi $=[\pi/4, \pi/2, 3\pi/4, \pi]$ and $||f'(x) - s'_N(x)||_\infty$ measured at the interpolated points xi and at the sample points x) against the mesh size h. Tabulate your results as follows:

h	2^{-4}	2^{-5}	2^{-6}
$\|\|f(\text{xi}) - s_N(\text{xi})\|\|_\infty$			
$\|\|f'(\text{xi}) - s'_N(\text{xi})\|\|_\infty$			
$\|\|f'(\text{x}) - s'_N(\text{x})\|\|_\infty$ at sample points x			

What is the rate of convergence of these piecewise cubic spline interpolation and derivative approximation? Does this agree with the theory?

5. **Chebychev Points:** The so-called Chebychev points

$$x_j = \cos(j\pi/N), \qquad j = 0, 1, \ldots, N,$$

which are the points at which the Chebychev polynomials take on extreme values in $[-1, 1]$ (check this), are commonly used in spectral approximations for non-periodic functions. Derive the density function $\rho_N(x)$ associated with these points? Using this function can you predict the performance of polynomial approximation sampled at these points. Use your Nddiff routine developed in Q2 to interpolate the function $f(x) = (1 - x)^{1/3}$ at the points xi=-1:0.005:1 by sampling the function at the Chebychev points for $N = 17$, and at the Chebychev roots for $N = 17$. Plot the interpolants and the errors at the xi. How do these interpolants and their errors compare? Now repeat this experiment with $g(x) = (4 - x)^{1/3}$. Explain the difference between the interpolatns for f and g.