1: THE WAVE EQUATION

(A) FROM A CONSERVATION LAW

- CONSIDER THE CONSERVATION OF MASS OF A NONDIFFUSIVE TRACER IN 1D

$$\frac{\partial}{\partial t}(\rho c) + \frac{\partial}{\partial x}(\rho c \bar{V}) = 0 \qquad (1)$$

- ρ = DENSITY
- c = CONCENTRATION OF THE TRACER
- $\bar{V}$ = PARTICLE VELOCITY

- CONTINUITY EQUATION

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \bar{V}) = 0 \qquad (2)$$

EXPAND (1)

$$\underbrace{c\left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \bar{V})\right\}}_{\overset{(2)}{=}\,0} + \rho\left\{\frac{\partial c}{\partial t} + \bar{V}\frac{\partial c}{\partial x}\right\} = 0$$

$$\therefore \quad \boxed{\frac{\partial c}{\partial t} + \bar{V}\frac{\partial c}{\partial x} = 0}$$

(B) SOLUTION BY CHARACTERISTICS
'FOLLOW THE PARTICLES'

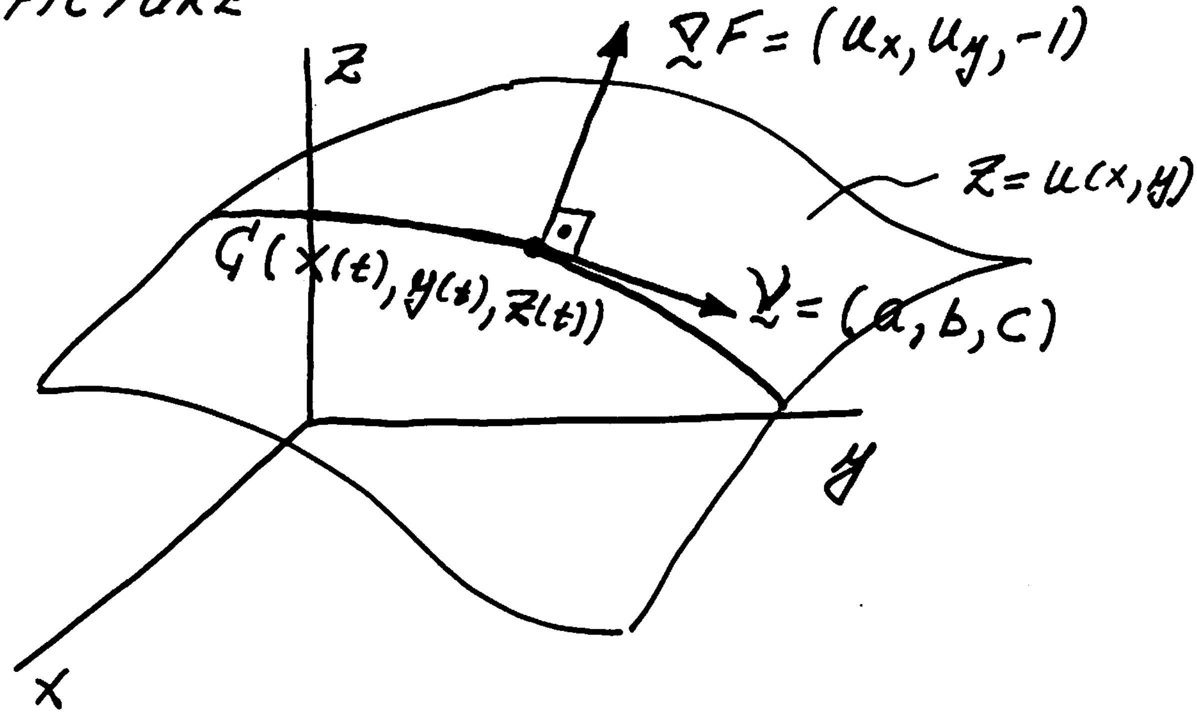
$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \qquad (*)$$

- SOLUTION IS A SURFACE

$$z = u(x,y)$$

OR $\quad F(x,y,z) = u(x,y) - z$

- PICTURE



$$(*) \Rightarrow \quad \underbrace{(a,b,c)}_{\underset{\sim}{V}} \cdot \underbrace{(u_x, u_y, -1)}_{\nabla F} = 0$$

$$\underset{\sim}{V} \cdot \nabla F = 0$$

IDEA: SURFACE $u(x,y)$ IS TANGENT TO $\underset{\sim}{V} = (a,b,c)$

• C TANGENT TO (a, b, c)

$$\Rightarrow \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt} \right) = k(a, b, c)$$

CHARACTERISTIC EQUATIONS

$$\boxed{\begin{array}{c} \dfrac{dx}{dt} = ka \; ; \quad \dfrac{dy}{dt} = kb \; ; \quad \dfrac{dz}{dt} = kc \\ \\ \text{OR} \\ \\ \dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{dz}{c} = k\,dt \end{array}}$$

EXAMPLE: WAVE EQUATION WITH CONSTANT VELOCITY C:

$$u_t + c\,u_x = 0 \; ; \qquad u(x, 0) = u_0(x)$$

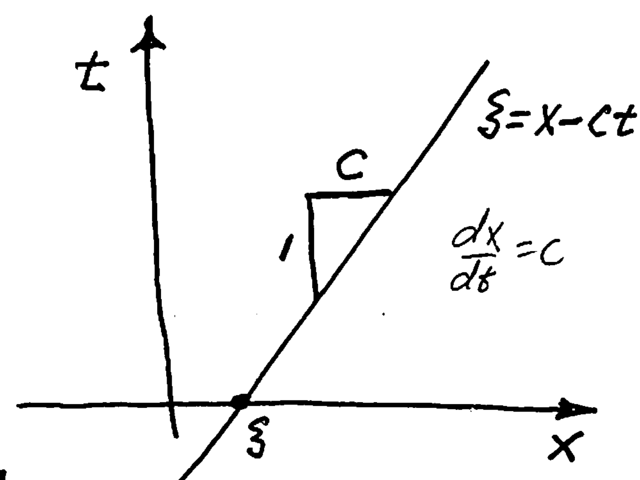
$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}$$

$\dfrac{dt}{1} = \dfrac{dx}{c} \Rightarrow \quad x - ct = \text{CONST} = \xi$

$\dfrac{du}{0} = \dfrac{dt}{1} \Rightarrow \quad u = \text{CONST} = \beta$

$t=0: \quad u(x,0) = u_0(x) = \beta$

$\qquad x - c.0 = x = \xi$



$$\boxed{u(x,t) = u_0(\xi) = u_0(x - ct)}$$

(B) FOURIER ANALYSIS OF SEMIDISCRETIZATIONS

DEFINE THE DISCRETE FOURIER TRANSFORM PAIR:

$$\bar{u}(\omega) = h \sum_{n=-\infty}^{\infty} e^{-i\omega x_n} u_n \leftrightarrow u_n = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{i\omega x_n} \bar{u}(\omega)\, d\omega$$

$$\mathcal{F}\{u_{n+1}\} = h \sum_{n=-\infty}^{\infty} e^{-i\omega x_n} u_{n+1} \qquad m = n+1 \quad n = m-1$$

$$= h \sum_{m=-\infty}^{\infty} e^{-i\omega h(m-1)} u_m$$

$$= e^{i\omega h}\, h \sum_{m=-\infty}^{\infty} e^{-i\omega x_m} u_m$$

$$= e^{i\omega h}\, \bar{u}(\omega)$$

$$\boxed{\mathcal{F}\{u_{n\pm p}\} = e^{\pm i\omega p h}\, \bar{u}(\omega)}$$

(C) SOLUTION BY FOURIER TRANSFORM (FT)

$$u_t + c\,u_x = 0; \quad u(x,0) = u_0(x) \qquad (1)$$

DEFINE THE FT PAIR:

$$\hat{u}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} u(x)\,dx; \quad u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\omega x}\hat{u}(\omega)\,d\omega$$

$$\widehat{\frac{\partial u}{\partial x}} = \int_{-\infty}^{\infty} e^{-i\omega x} u'(x)\,dx$$

$$= u e^{-i\omega x}\Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} e^{-i\omega x} u(x)\,dx$$

$$= +i\omega\,\hat{u}(\omega).$$

$$FT(1) \Rightarrow \boxed{\frac{d\hat{u}}{dt} = -i\omega c\,\hat{u}}; \quad \hat{u}(\omega,0) = \hat{u}_0(\omega)$$

$$\hat{u}(\omega,t) = e^{-i\omega c t}\,\hat{u}_0(\omega)$$

$$\therefore \quad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\omega(x-ct)}\,\hat{u}_0(\omega)\,d\omega$$

$$= u_0(x-ct)$$

(D) CONSERVATION OF ENERGY FOR WAVE EQ

DEFINE

$$\mathcal{E}_u(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 dx = \|u(t)\|_2^2$$

- $$0 = \int_{-\infty}^{\infty} u(u_t - cu_x)\, dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t} u^2 - \frac{c}{2}\frac{\partial}{\partial x} u^2 \, dx$$
$$= \frac{1}{2}\frac{\partial}{\partial t}\mathcal{E}_u(t) - \frac{c}{2}\left[\cancel{u}\right]_{-\infty}^{\infty}$$

$\therefore \quad \dot{\mathcal{E}}_u = 0 \Rightarrow \boxed{\mathcal{E}_u = \text{CONST}}$

- USING FT:

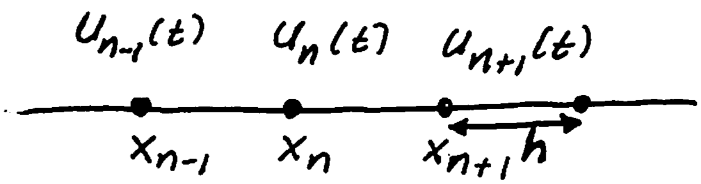
$$\mathcal{E}_u(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(\omega,t)|^2 d\omega \quad \text{PARSEVAL'S THM}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |e^{-i\omega ct}\hat{u}_0(\omega)|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 d\omega = \text{CONST}$$

NOTE: $|\hat{u}(\omega,t)| = |\hat{u}_0(\omega)| \Rightarrow$ THE SPECTRAL DISTRIBUTION OF ENERGY IS TIME INDEPENDENT

2. APPROXIMATE SOLUTIONS TO THE WAVE EQ

$$u_t + c\,u_x = 0$$

(A) SEMIDISCRETIZATIONS:



$u(x_n, t) \simeq u_n(t)$

$x_n = nh$

CENTRAL DIFFERENCE:

$$\dot{u}_n = -c\left(\frac{u_{n+1} - u_{n-1}}{2h}\right) = -c\,\frac{\partial u_n}{\partial x} + O(h^2)$$

$$= A_c \cdot u_n$$

FORWARD DIFFERENCE:

$$\dot{u}_n = -c\left(\frac{u_{n+1} - u_n}{h}\right) = -c\,\frac{\partial u_n}{\partial x} + O(h)$$

$$= A_f \cdot u_n$$

BACKWARD DIFFERENCE

$$\dot{u}_n = -c\left(\frac{u_n - u_{n-1}}{h}\right) = -c\,\frac{\partial u_n}{\partial x} + O(h)$$

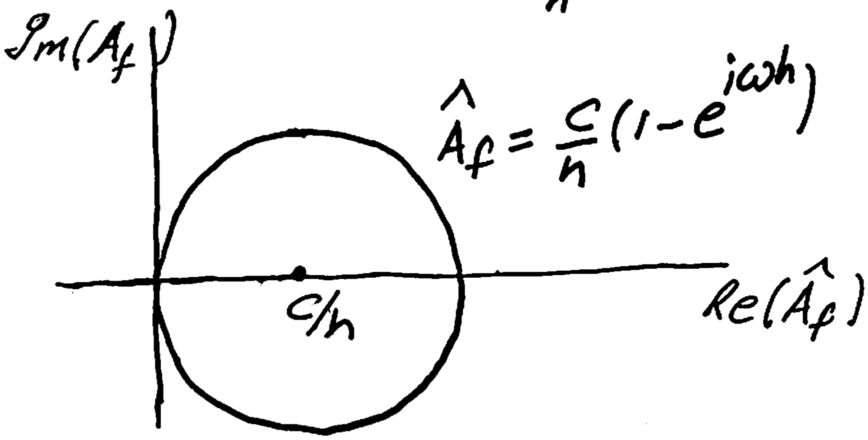
$$= A_B \cdot u_n$$

ANALYSIS OF THE FORWARD DIFFERENCE SCHEME

$$\dot{u}_n = A_f \cdot u_n = -\frac{c}{h}(u_{n+1} - u_n)$$

$$\dot{\hat{u}}(\omega, t) = \underbrace{-\frac{c}{h}(e^{i\omega h} - 1)}\,\hat{u}(\omega, t)$$

$$\hat{A}_f(\omega) = -\frac{c}{h}(e^{i\omega h} - 1)$$ EIGENVALUES



$$u_n(t) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \hat{u}_0(\omega)\, e^{Re(\hat{A}_f)\,t}\, e^{i\{\omega x_n + Im(\hat{A}_f)\,t\}}\, d\omega$$

POSITIVE REAL
PART - UNSTABLE

ANOTHER WAY TO EXPLAIN THE INSTABILITY:

- $$\dot{u}_n = -\frac{c}{h}(u_{n+1} - u_n)$$
  $$= -\frac{c}{h}\left(u_n + h u_n' + \frac{h^2}{2} u_n'' + \cdots - u_n\right)$$
  $$\simeq -c u_n' - \frac{ch}{2} u_n''$$

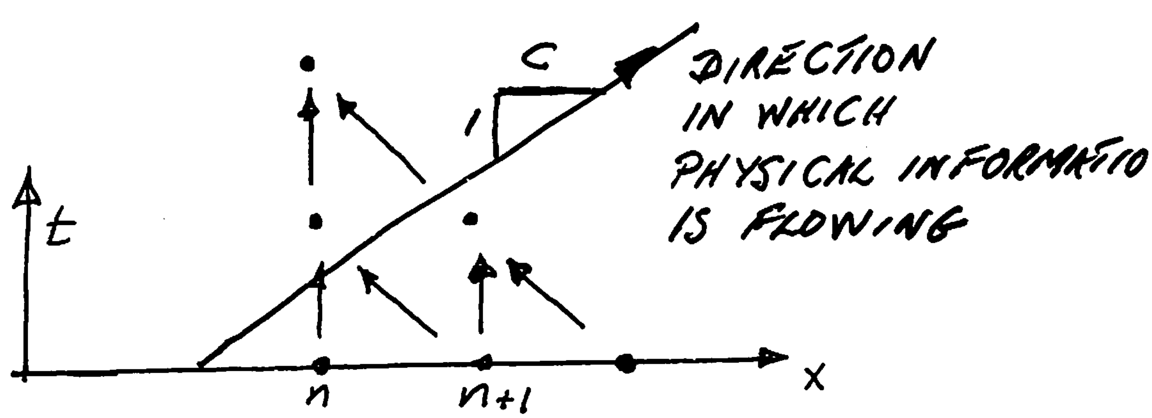
THIS SCHEME TRIES TO MODEL THE WAVE EQ

BY PROVIDING THE SOLUTION TO THE

BACWARD CONVECTIVE HEAT EQ

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{ch}{2}\frac{\partial^2 u}{\partial x^2}$$ ← BAD IDEA

- FLOW OF INFORMATION — WRONG WAY!



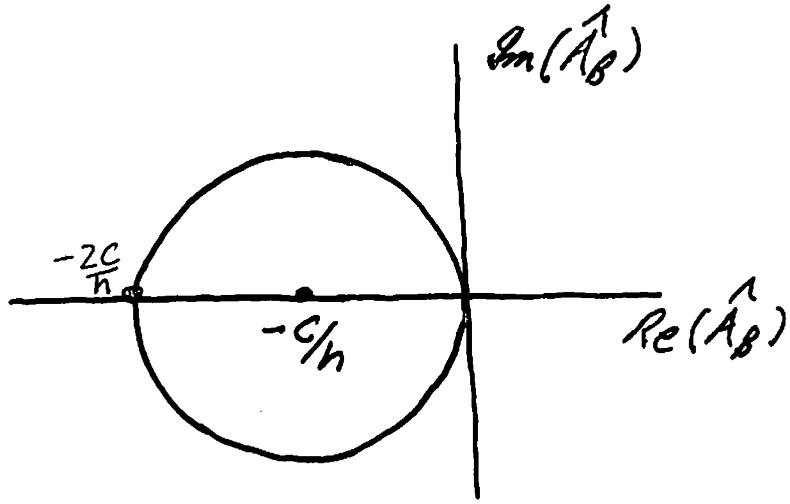
- 'DOWNWIND DIFFERENCING'

ANALYSIS OF THE BACWARD DIFFERENCE SCHEME

$$\dot{u}_n = A_B \cdot u_n = -\frac{c}{h}(u_n - u_{n-1})$$

$$\dot{\hat{u}}(\omega,t) = \underbrace{-\frac{c}{h}(1-e^{-i\omega h})}\,\hat{u}(\omega,t)$$

$$\hat{A}_B(\omega) = \frac{c}{h}(e^{-i\omega h}-1) \quad \text{EIGENVALUES}$$



EULER + UPWIND

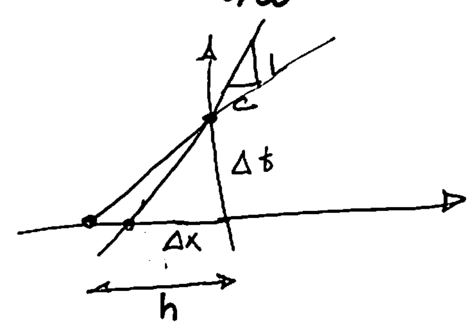
$$-2 < \frac{-2C\Delta t}{h}$$

$$\therefore \quad \Delta t < \frac{h}{c}$$

$$\frac{\Delta x}{\Delta t} = c < \frac{h}{\Delta t}$$

$$u_n(t) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \hat{u}_0(\omega)\, e^{Re(\hat{A}_B)t}\, e^{i(\omega x_n + Im(\hat{A}_B)t)}\, d\omega$$

NEGATIVE REAL PART
⇒ DAMPING



PHYSICAL EXPLANATION FOR THE DAMPING

- $$\dot{u}_n = -\frac{c}{h}\left(u_n - u_{n-1}\right)$$
$$= -\frac{c}{h}\left(u_n - u_n + h u_n' - \frac{h^2}{2} u_n'' + \dots\right)$$
$$\cong -c u_n' + \frac{ch}{2} u_n''$$

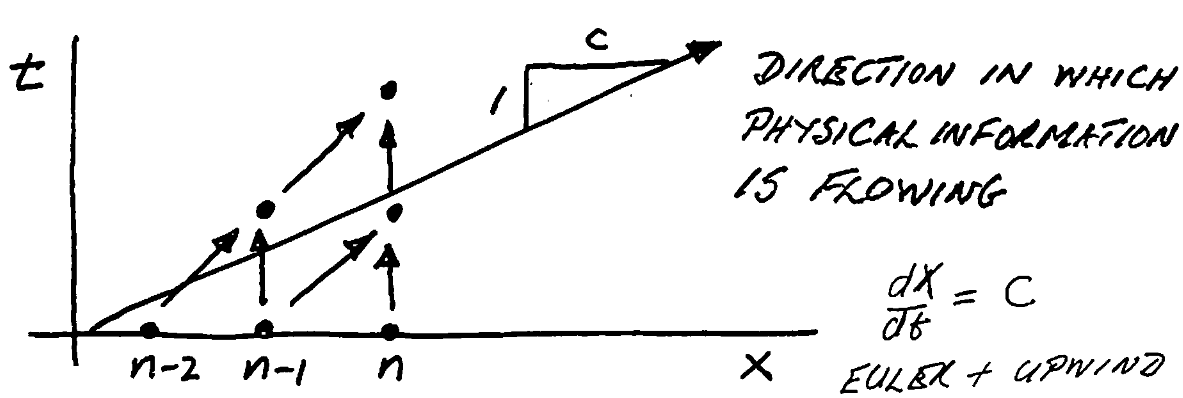
THUS THE BACKWARD DIFFERENCE SCHEME MODELS THE WAVE EQ BY PROVIDING THE SOLUTION TO THE CONVECTIVE HEAT EQ:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{ch}{2}\frac{\partial^2 u}{\partial x^2}$$

THIS TERM ADDS A LITTLE NUMERICAL DIFFUSION

- THE NUMERICAL DIFFUSION ENHANCES STABILITY BUT ENERGY IS NO LONGER CONSERVED.

- FLOW OF INFORMATION – THE CORRECT WAY!

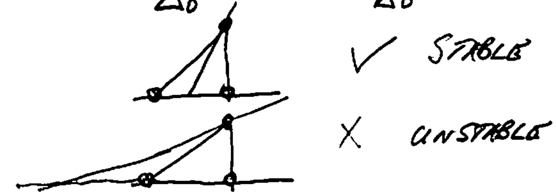


$$\frac{dx}{dt} = C$$

EULER + UPWIND

$$\frac{\Delta x}{\Delta t} = C < \frac{h}{\Delta t}$$

- (UPWIND DIFFERENCING)



FT ANALYSIS OF CENTRAL DIFFERENCE SCHEME

$$\dot{u}_n = A_c \cdot u_n = -\frac{c}{2h}(u_{n+1} - u_{n-1})$$

$$\dot{\hat{u}} = -\frac{c}{2h}\left(e^{i\omega h} - e^{-i\omega h}\right)\hat{u}$$

$$= \underbrace{-\frac{ci}{h}\sin\omega h}_{\hat{A}_c(\omega)}\,\hat{u}(\omega,t)$$

$\hat{A}_c(\omega)$ EIGENVALUE

NOTE: $\hat{A}_c(\omega) = -\frac{ci}{h}\sin\omega h \overset{h\to 0}{\sim} -i\omega c$ WAVE EQ EIGENVALUES

ODE $\dot{\hat{u}} = \hat{A}_c(\omega)\hat{u}$; $\hat{u}(\omega,0) = \hat{u}_0(\omega)$

SOLUTION: $\hat{u}(\omega,t) = e^{\hat{A}_c(\omega)t}\,\hat{u}_0(\omega)$

INVERT FT:

$$u_n(t) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{i\omega\left(x_n - c\frac{\sin\omega h}{\omega h}t\right)}\hat{u}_0(\omega)\,d\omega$$

THE QUANTITY $c_c^*(\omega) = c\frac{\sin\omega h}{\omega h}$ IS KNOWN AS THE PHASE VELOCITY

IS THE CENTRAL DIFFERENCE SCHEME CONSERVATIVE?

$$\mathcal{E}_{u_n}(t) = \|u_n\|_2^2 = h \sum_{n=-\infty}^{\infty} |u_n|^2$$

$$= \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} |\hat{u}(\omega,t)|^2 \, d\omega \quad \text{BY PARSEVAL'S THM}$$

$$= \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} | e^{\hat{A}_c(\omega)t} \hat{u}_0(\omega) |^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \underbrace{e^{2 Re(\hat{A}_c(\omega))t}}_{1} \underbrace{|e^{i(\omega x_n + Im\, \hat{A}_c(\omega) t)}|^2}_{1} |\hat{u}_0(\omega)| \, d\omega$$

SINCE $\hat{A}_c(\omega) = -\frac{ic}{h} \sin(\omega h) \Rightarrow Re(\hat{A}_c) = 0$

$\therefore \quad \boxed{\mathcal{E}_{u_n}(t) = \text{CONST}}$

THE CAUSE OF THE NUMERICAL DISPERSION

$$
\begin{aligned}
\dot{u}_n &= -\frac{c}{2h}\left(u_{n+1}-u_{n-1}\right) \\
&= -\frac{c}{2h}\left\{u_n + h u_n' + \frac{h^2}{2}u_n'' + \frac{h^3}{6}u_n''' + \cdots - u_n + h u_n' - \frac{h^2}{2}u_n'' + \frac{h^3}{6}u_n''' + \cdots\right\} \\
&= -c u_n' - \underbrace{\frac{ch^2}{6}u_n'''}_{\text{A DISPERSIVE TERM}} + \cdots
\end{aligned}
$$

THUS THE CENTRAL DIFFERENCE SCHEME 'MODELS' THE WAVE EQ BY PROVIDING THE SOLUTION TO THE PDE

$$
\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \underbrace{\frac{ch^2}{6}\frac{\partial^3 u}{\partial x^3}}_{\text{SMALL IF } h \ll 1} = 0
$$

LARGE TIME BEHAVIOUR OF $u_n(t)$: $t \to \infty$

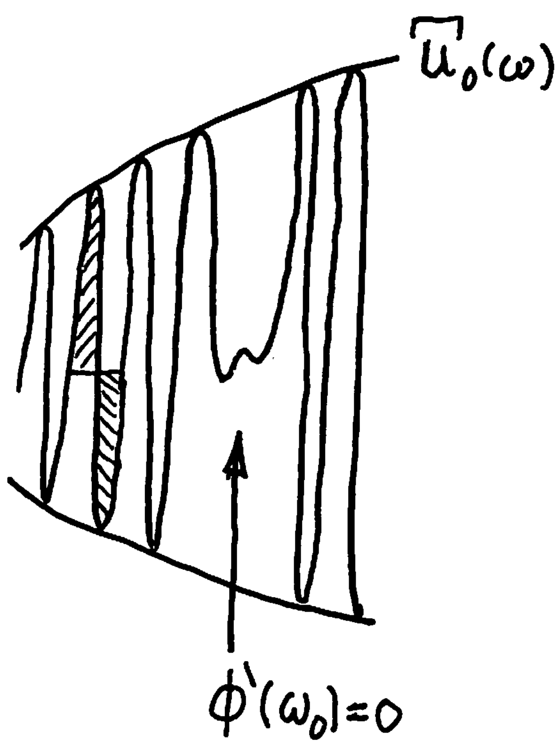
$$u_n(t) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{i\left(\omega \frac{x}{t} - \frac{c \sin \omega h}{h}\right) t} \,\overline{u}_0(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{i\phi(\omega) t} \,\overline{u}_0(\omega)\, d\omega \qquad (*)$$

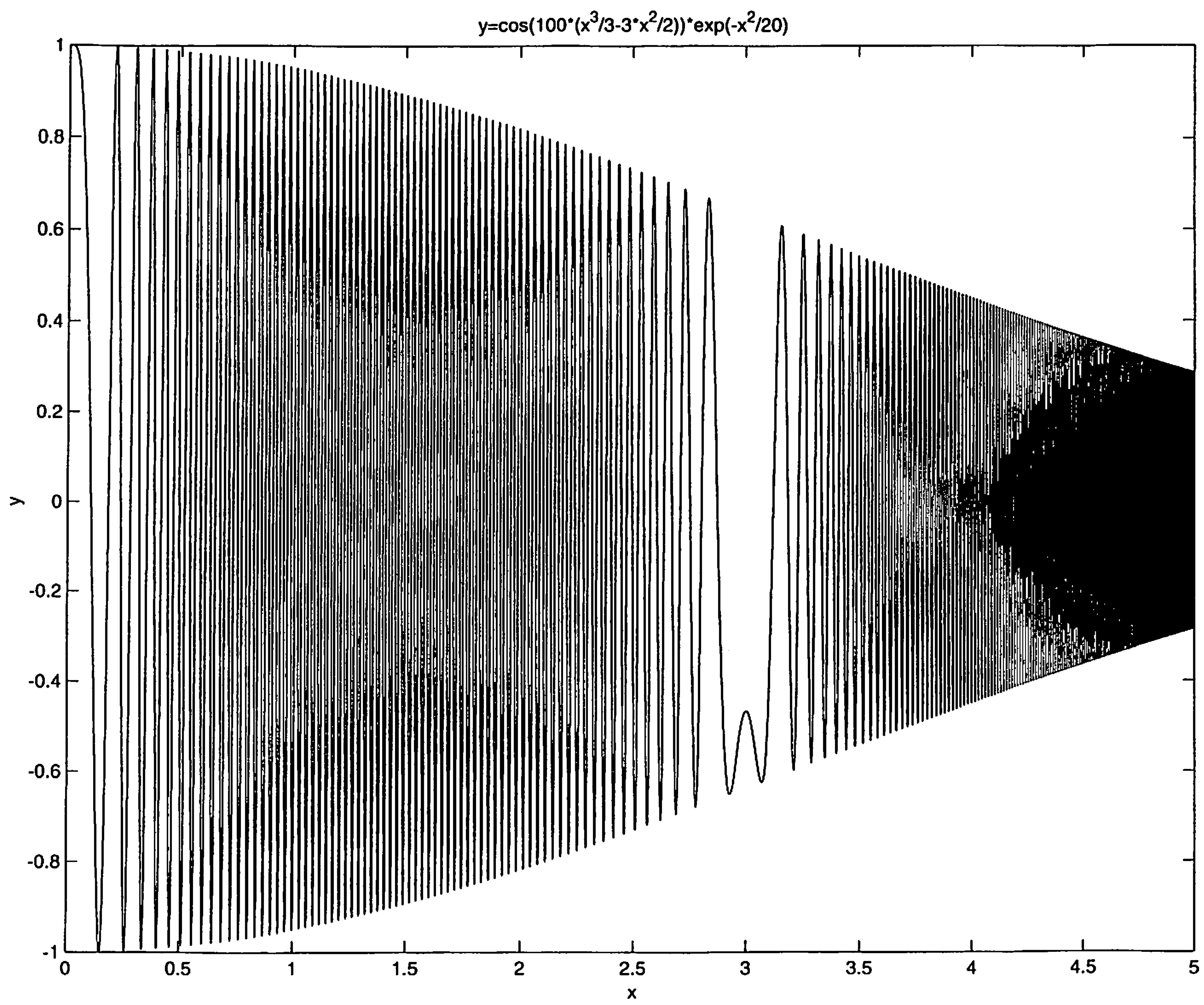
LOCAL ANALYSIS ABOUT ANY POINT $\omega_0 \in [-\pi/h, \pi/h]$

$$\overline{u}_0(\omega)\, e^{i\phi(\omega)t} = \overline{u}_0(\omega)\, e^{i\{\phi(\omega_0)t + \phi'(\omega_0)(\omega-\omega_0)t + \cdots\}}$$

$$= \overline{u}_0(\omega)\, e^{i\phi_0 t}\, e^{i\phi_0'(\omega-\omega_0)t}\, e^{i\phi_0'' \frac{(\omega-\omega_0)^2}{2} t} + \cdots$$



IDEA: THE DOMINANT CONTRIBUTION TO THE INTEGRAL (*) COMES FROM POINTS FOR WHICH $\phi'(\omega)=0$



y=cos(100*(x^3/3-3*x^2/2))*exp(-x^2/20)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
x

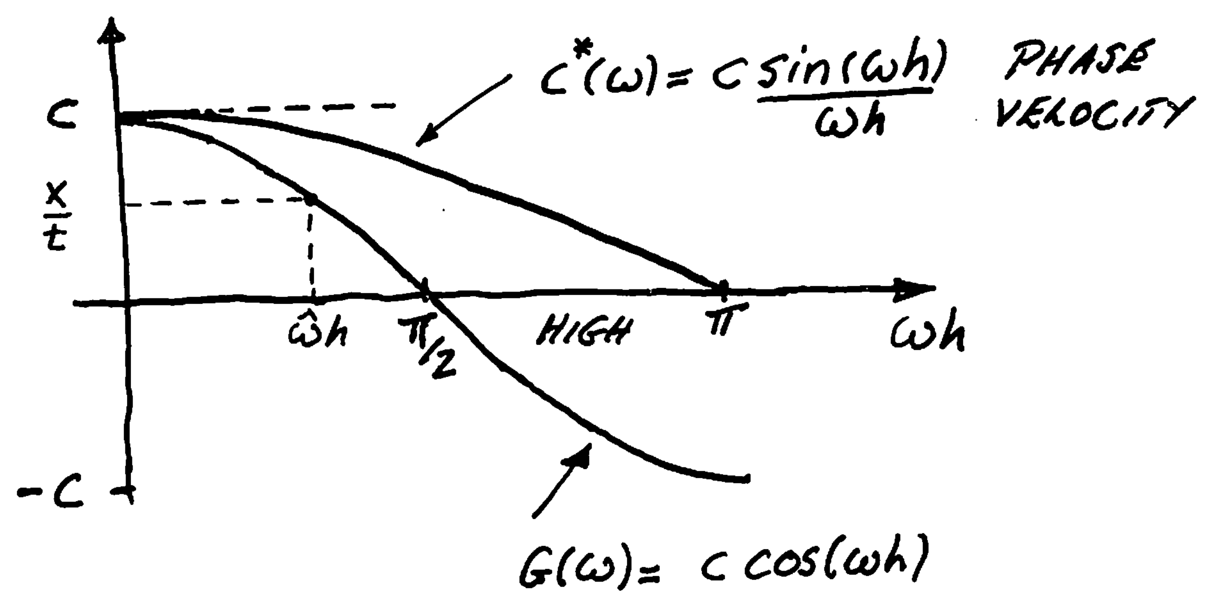
FOR OUR PROBLEM

$$\phi(\omega) = \omega \frac{x}{t} - c\,\frac{\sin(\omega h)}{h}$$

$$\phi'(\omega) = \frac{x}{t} - c\cos(\omega h) = 0$$

$$\Rightarrow \quad \boxed{x = c\cos(\omega h)\,t = G(\omega)\,t} \qquad (*)$$

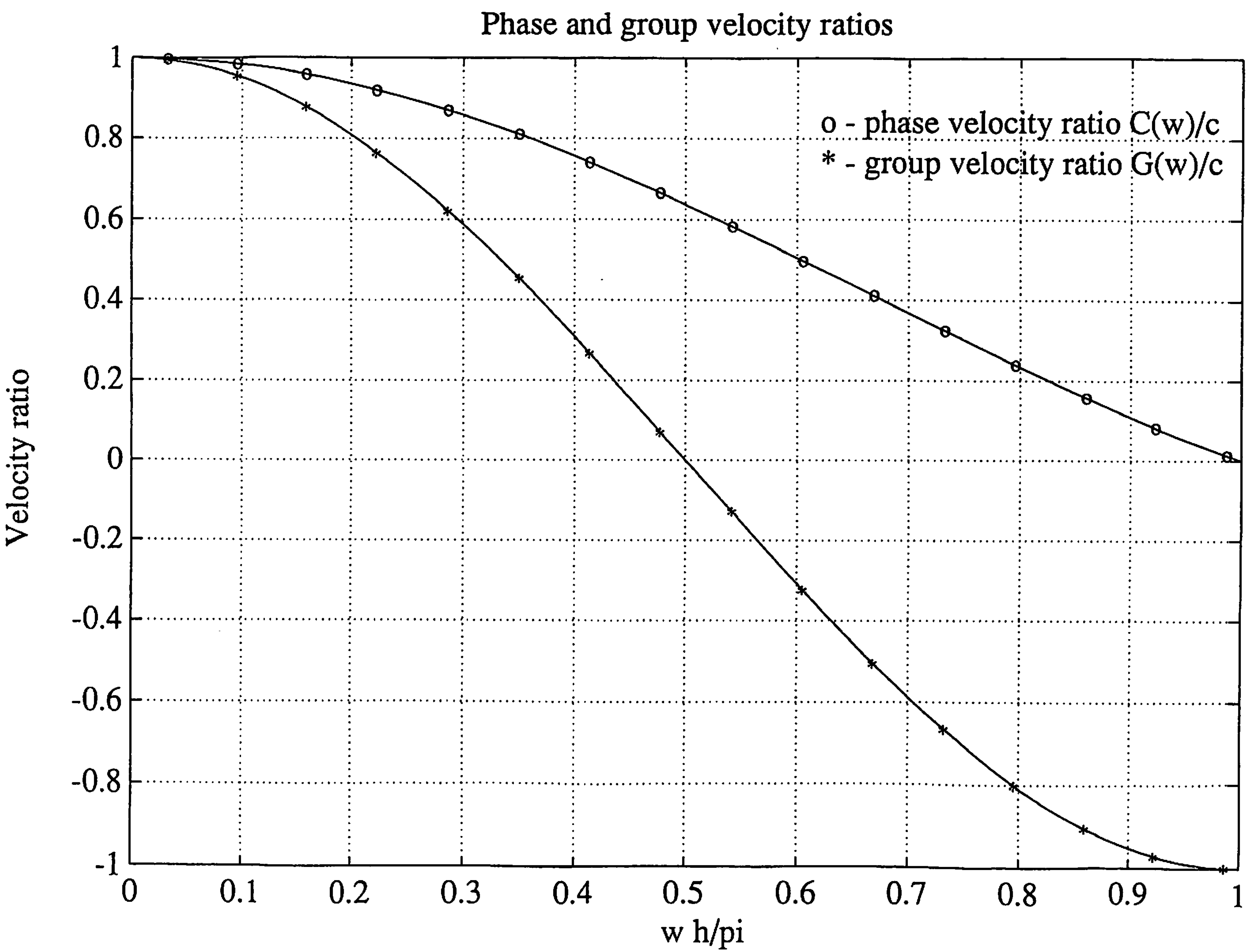
WHERE $G(\omega) = c\cos(\omega h)$ IS THE GROUP VELOCITY



INTERPRETATION OF (*):

EACH RAY $\frac{x_n}{t}$ IS ASSOCIATED WITH A GROUP OF FREQUENCIES IN THE NEIGHBOURHOOD OF

$$\hat{\omega}h = \cos^{-1}(x_n/ct)$$

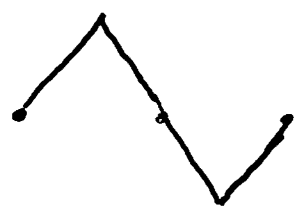

Phase and group velocity ratios
o - phase velocity ratio C(w)/c
* - group velocity ratio G(w)/c
Velocity ratio
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
w h/pi
Figure 1

19

FREQUENCY – WAVELENGTH CATALOGUE

$$e^{i\omega(x_n+\lambda)} = e^{i\omega x_n} \qquad \lambda = \frac{2\pi}{\omega}$$

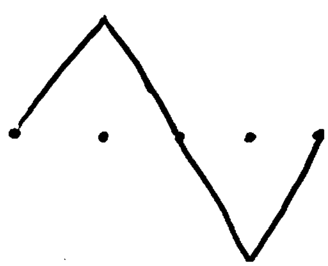
HIGH



$\lambda = 2h$

$\omega h = \pi$ NYQUIST FREQUENCY

MEDIUM



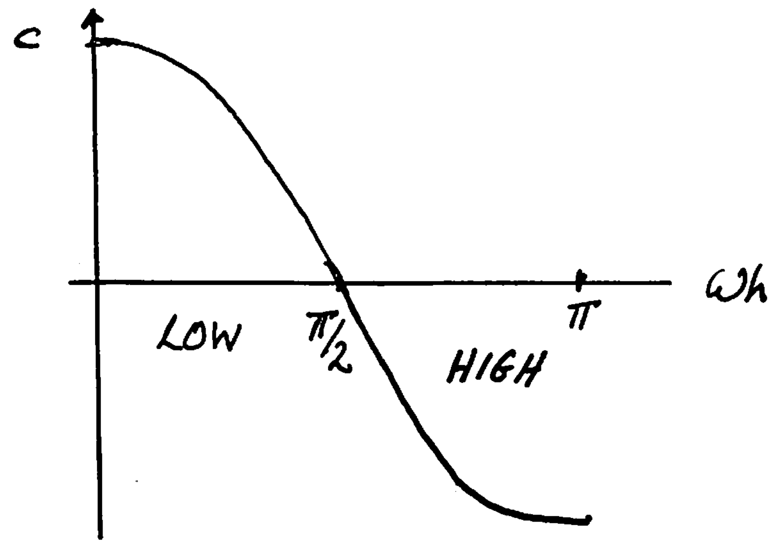
$\lambda = 4h$

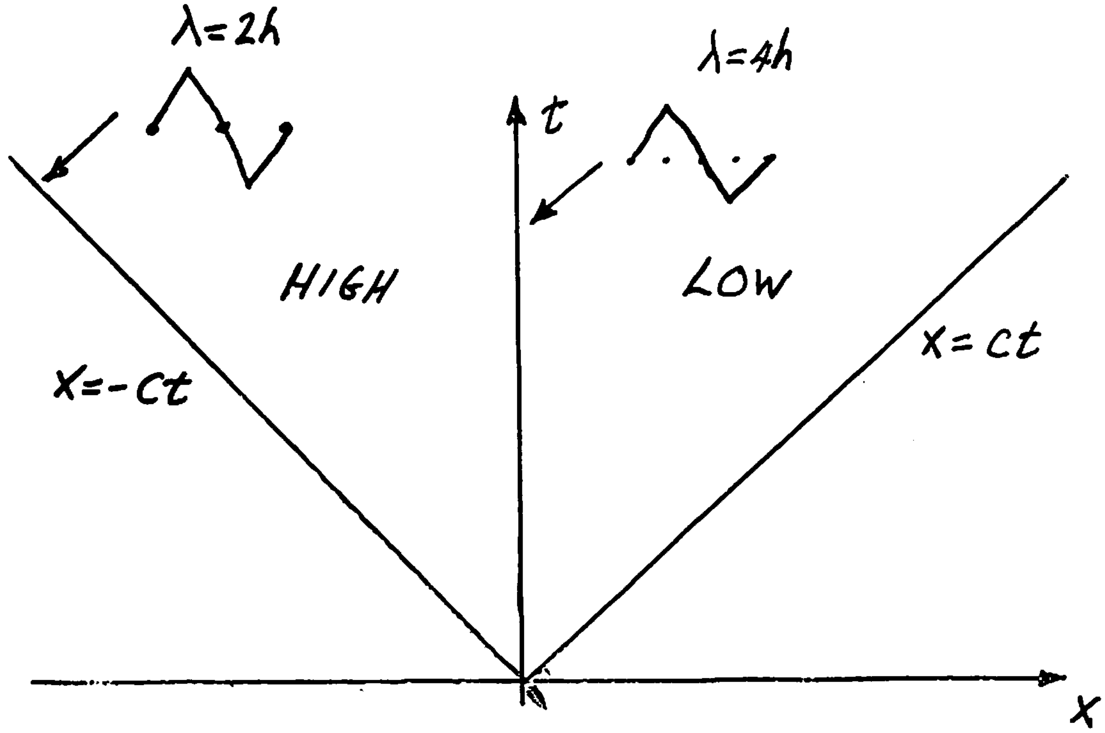
$\omega h = \pi/2$

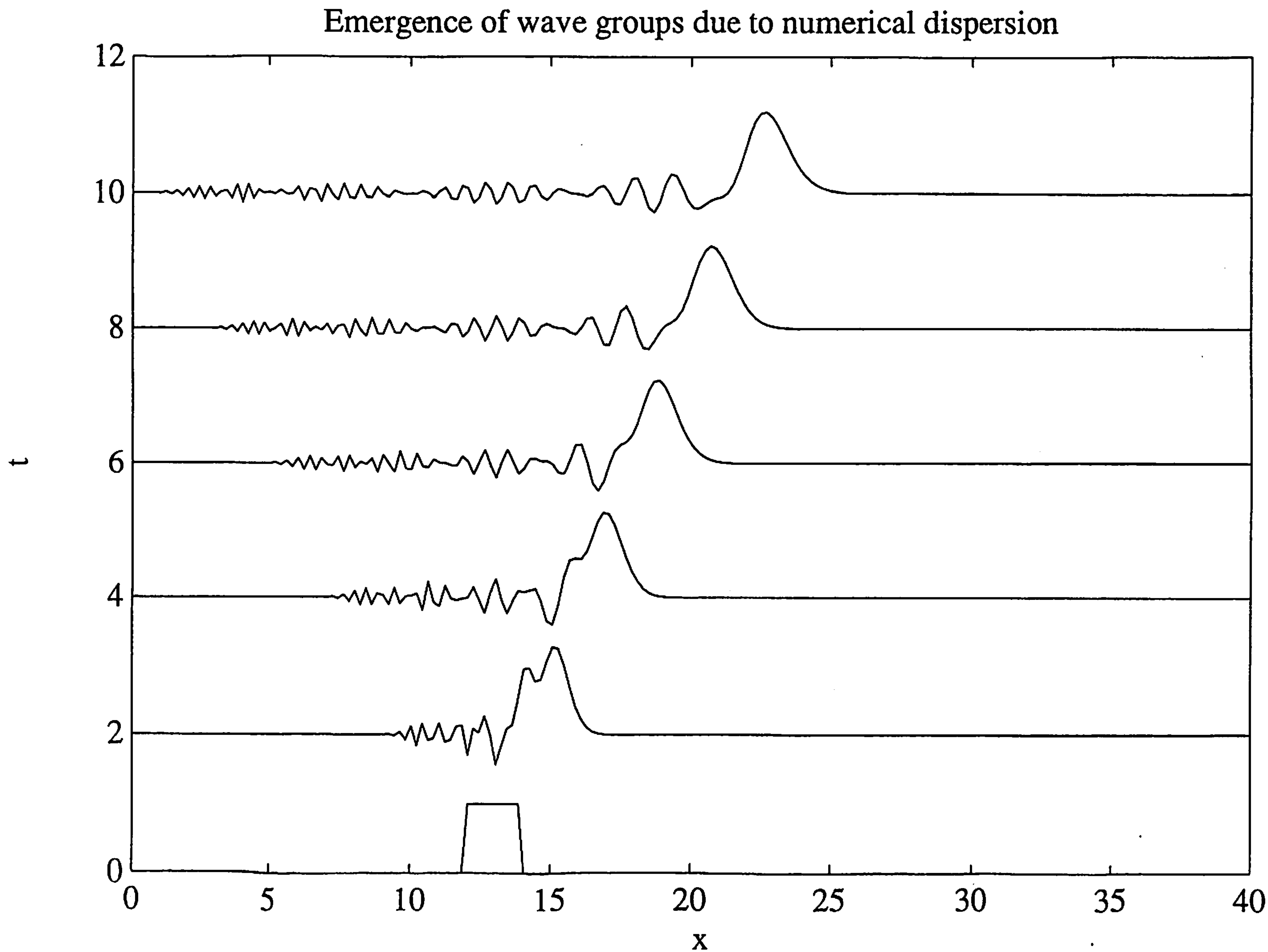
LOW

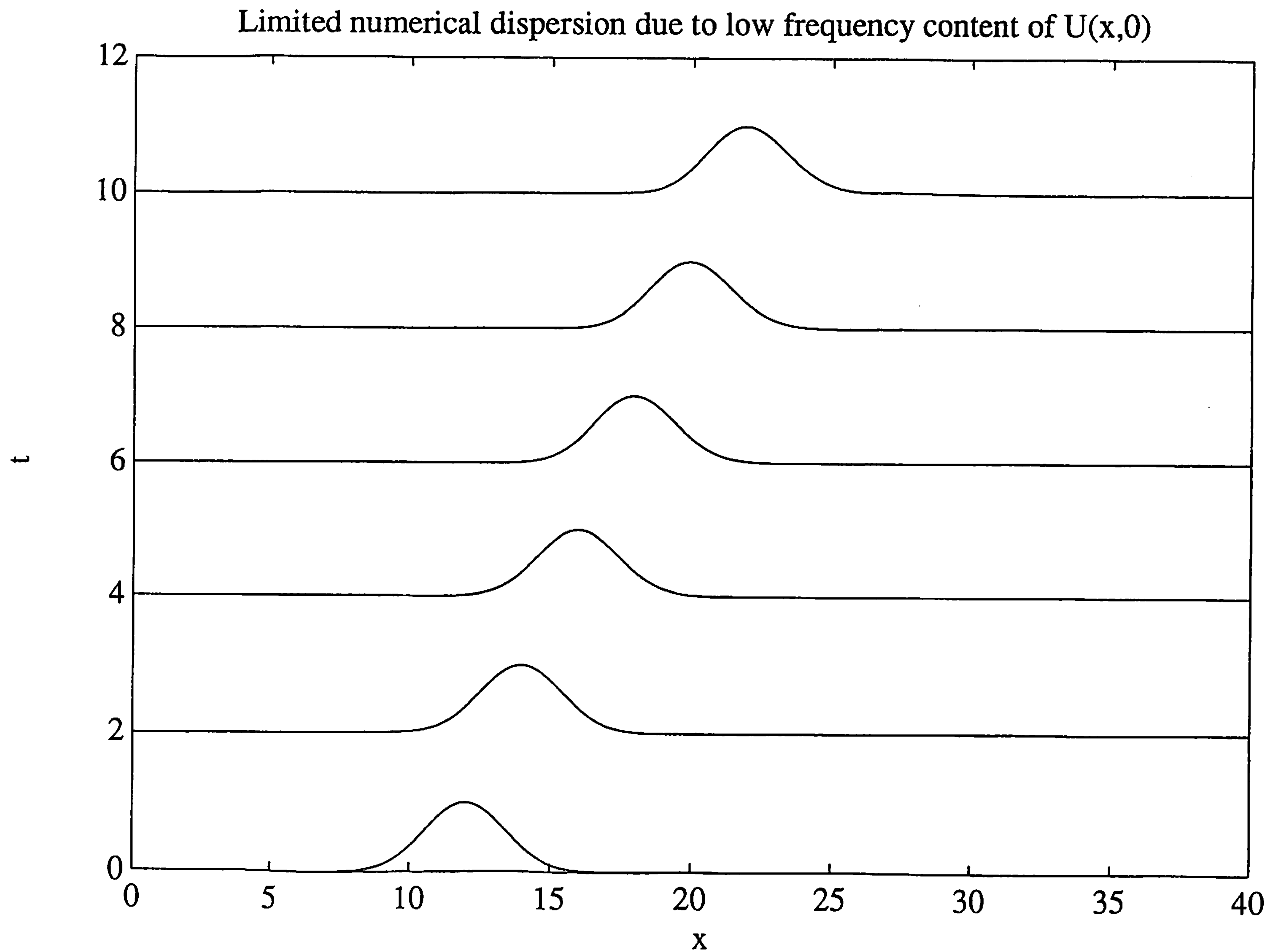
$\omega h < \pi/2$

SPATIAL PICTURE:






Emergence of wave groups due to numerical dispersion
12
10
8
6
4
2
0
t
0
5
10
15
20
25
30
35
40
x
Figure 2


Limited numerical dispersion due to low frequency content of U(x,0)
12
10
8
6
4
2
0
t
0
5
10
15
20
25
30
35
40
x
Figure 3

23