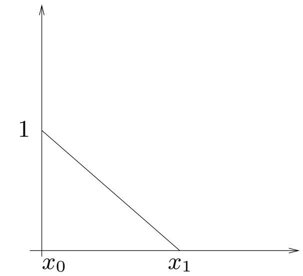
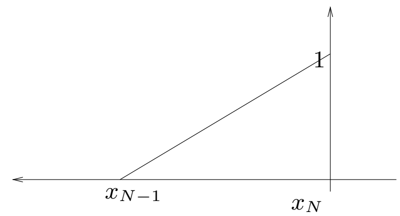
$$N_0^1(x) = \begin{cases} \left(\frac{x_1-x}{x_1-x_0}\right) & x \in [x_0, x_1] \\ 0 & x \notin [x_0, x_1] \end{cases}$$



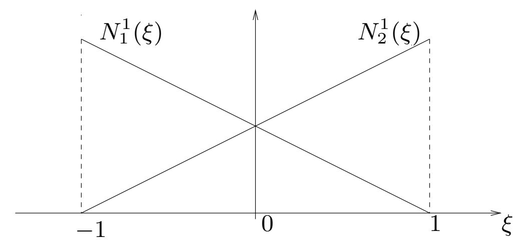
$$N_n^1(x) = \begin{cases} 0 & x \notin [x_{N-1}, x_N] \\ \left(\frac{x-x_{N-1}}{x_N-x_{N-1}}\right) & x \in [x_{N-1}, x_N] \end{cases}$$



Then $p_{i,N}(x) = \sum_{i=0}^{N} f_i N_i^1(x) \approx f(x)$. We notice that $N_i^1(x_j) = \delta_{ij}$ so that the basis functions are zero outside the interval (x_{i-1}, x_{i+1}) – we say that such *basis functions have local support*.

Representation on a canonical interval:

Sometimes it is more convenient to perform calculations by representing the piecewise linear basis functions on a canonical interval: $[-1, 1]$. On the canonical interval the basis functions assume the form.:



Canonical Interval

$$N_1^1(\xi) = \frac{1}{2}(1-\xi) \qquad N_2^1(\xi) = \frac{1}{2}(1+\xi)$$

or $$N_a^1(\xi) = \frac{1}{2}(1+\xi_a \xi) \qquad \xi_1 = -1 \qquad \xi_2 = +1$$

Note: $N_a^1(\xi_b) = \delta_{ab}$ and $x(\xi) = \sum_{a=1}^{2} x_a N_a^1(\xi)$

Error involved: Recall $e_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$ for polynomial interpolants.

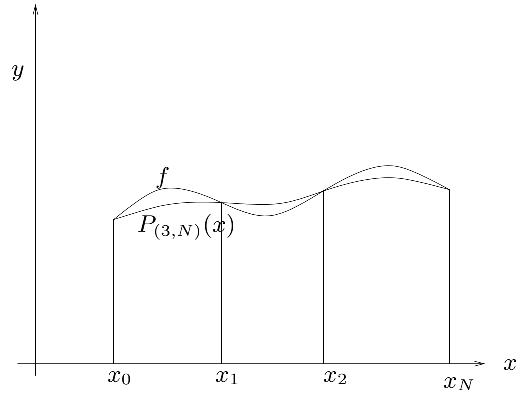
$$\max_{x\in[x_i,x_{i+1}]} |e_1(x)|$$
$$= \max_{x\in[x_i,x_{i+1}]} \left| \frac{f''(s)}{2}(x-x_i)(x-x_{i+1}) \right| \leq \frac{1}{2}||f''||_\infty \max_{x\in[x_i,x_{i+1}]} |(x-x_i)(x-x_{i+1})| \leq \frac{h^2}{8}||f''||_\infty.$$

Using:

$$w(x) = (x-x_i)(x-x_{i+1}) = x^2 - (x_i + x_{i+1})x + x_i x_{i+1}$$

$$w'(x) = 2x - (x_i + x_{i+1}) = 0 \Rightarrow \quad x = \frac{(x_i + x_{i+1})}{2} \ \& \ w\left(\frac{x_i + x_{i+1}}{2}\right) = \left(\frac{x_{i+1} - x_i}{2}\right)\left(\frac{x_i - x_{i+1}}{2}\right)$$

2. **Piecewise quadratic interpolation**:



Degree of freedom analysis:

N Subintervals 3 coefficients for quadratic
$3N$ DOF

Constraints

(1) Continuity at interior points
Continuity of derivative at interior points $\Big\} \Rightarrow 2(N-1)$ constraints

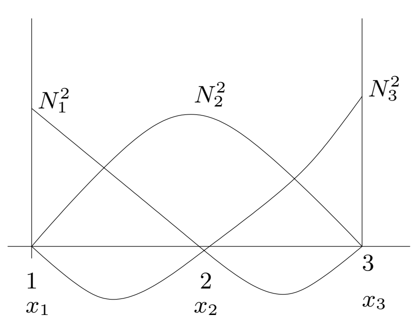
$$\begin{aligned}\text{Remaining DOF } = 3N - 2(N-1) = N+2 &= (N+1)+1 \\ &= \text{function values at } N+1 \text{ nodes} \\ &\quad\ \text{and 1 extra condition (?)}\end{aligned}$$

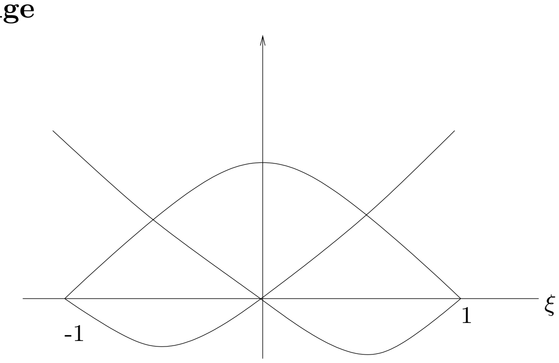
(2) Continuity at interior nodes $\Rightarrow N-1$

$$\begin{aligned}\text{Remaining DOF } = 3N - (N-1) = 2N+1 &= (N+1)+N \\ &= \text{function values at } (N+1) \text{ nodes} \\ &\quad\ +1 \text{ function value within each interval}\end{aligned}$$

These are called *quadratic Lagrange interpolants*.

Basis functions representation for quadratic Lagrange





On Canonical Interval

$$
\begin{aligned}
N_1^2(x) &= \frac{(x-x_2)}{(x_1-x_2)}\frac{(x-x_3)}{(x_1-x_3)} & \qquad N_1(\xi) = \frac{1}{2}\xi(\xi-1) \\
N_2^2(x) &= \frac{(x-x_1)}{(x_2-x_1)}\frac{(x-x_3)}{(x_2-x_3)} & \qquad N_2(\xi) = 1-\xi^2 \\
N_3^2(x) &= \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} & \qquad N_3(\xi) = \frac{1}{2}\xi(\xi+1)
\end{aligned}
$$

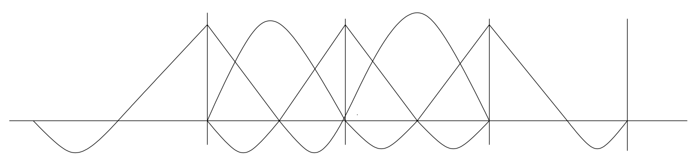
Note:

(1) $N_i^2(x_j) = \delta_{ij}$ • On a canonical interval $[-1,1]$.

(2) $\sum_{i=1}^{N} N_i(x) = 1$ $\quad N_1 + N_2 + N_3 = \frac{1}{2}\cancel{\xi^2} - \frac{1}{2}\cancel{\xi} + 1 - \cancel{\xi^2} + \frac{1}{2}\cancel{\xi^2} + \frac{1}{2}\cancel{\xi} = 1$

Must be true as we must be able to represent a constant function exactly. We can now obtain a global representation of the interpolants by numbering all the basis functions:

$f(x) \sim \sum_{i=0}^{n} f_i N_i^2(x)$



3. **Piecewise cubic interpolants**: $p_{3,N}(x)$

DOF Analysis: *FIGURE*

N intervals $\quad N+1$ nodes $\quad N-1$ internal nodes

4 unknown coefficients for interval

Schemes of constraint:

(1) $p_{3,N}(x)$ and $p'_{3,N}(x)$ are continuous at interior nodes.

$$
\begin{aligned}
\Rightarrow \text{ DOF } : 4N - 2(N-1) &= 2(N+1) \\
&\Rightarrow \text{ specify function value and} \\
&\qquad \text{its derivative at all } N+1 \text{ nodes}
\end{aligned}
$$

⇒ **Piecewise Cubic Hermite polynomials**

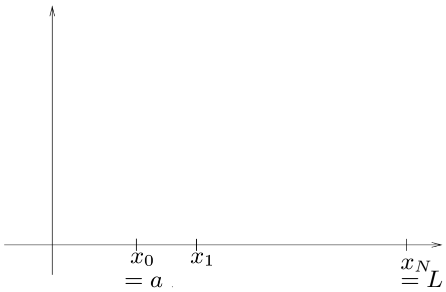
• Have to specify f & f' at all $N+1$ nodes!

(2) $p_{3,N}(x)$, $p'_{3,N}(x)$, $p''_{3,N}(x)$ are all continuous at interior.

$$
\begin{aligned}
\Rightarrow \text{ DOF } : 4N - 3(N-1) = N+3 &= \underbrace{N+1}+2 \\
&\Rightarrow \text{ specify } f \text{ at all } N+1 \text{ nodes} \\
&\qquad \text{and impose 2 EXTRA CONDITIONS}
\end{aligned}
$$

This is a *cubic spline*. The extra conditions are up to the user to prescribe depending on the application, e.g. $p_3''(x_0) = 0 = p_3''(x_N)$ which is called the natural spline.

Piecewise cubic interpolation:



Piecewise cubic Hermite polynomials:

DOF: $4N - 2(N-1) = 2(N+1) \Rightarrow$ prescribe $f(x_i)$ and $f'(x_i)$ at $i = 0, \ldots, N$.

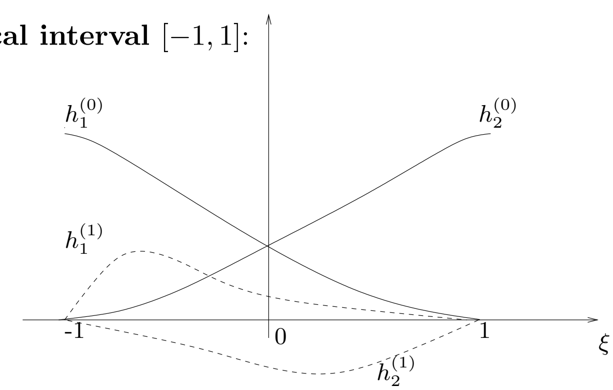
Presentation of f in terms of Hermite basis functions $h_i^{(0)}(x)$ and $h_i^{(1)}(x)$:

$$f(x) \approx h(x) = \sum_{i=0}^{N} f(x_i) h_i^{(0)}(x) + \sum_{i=0}^{N} f'(x_i) h_i^{(1)}(x)$$

where

$$\left.\begin{array}{rcr} h_i^{(0)}(x_j) = \delta_{ij} & \text{and} & \frac{d}{dx} h_i^{(1)}(x_j) = \delta_{ij} \\ \frac{d}{dx} h_i^{(0)}(x_j) = 0 & & h_i^{(1)}(x_j) = 0 \end{array}\right\} \qquad (1)$$

Constructing basis functions on the canonical interval $[-1, 1]$:



We use the linear Lagrange basis functions

$$N_a^1(\xi) = \frac{1}{2}(1+\xi_a \xi) : N_a^1(\xi_b) = \delta_{ab}, \qquad \xi_{a,b} = \pm 1$$

Let

$$\begin{aligned} h_1^{(0)}(\xi) &= (\alpha_1 \xi + \beta_1)[N_1^1(\xi)]^2; h_2^{(0)}(\xi) = (\alpha_2 \xi + \beta)[N_2^1(\xi)]^2 \\ h_1^{(1)}(\xi) &= (\gamma_1 \xi + \delta_1)[N_1^1(\xi)]^2; h_2^{(1)}(\xi) = (\gamma_2 \xi + \delta_2)[N_2^1(\xi)]^2 \end{aligned}$$

To find α_i, β_i, γ_i and δ_i we impose the conditions (1):

$$1 = \quad h_1^{(0)}(-1) = \beta_1 - \alpha_1 \quad 0 = h_1^{(0)'}(-1) = \alpha_1 1 + (\beta_1 - \alpha_1)\frac{2}{2}(1-(-1))\frac{(-1)}{2} = 2\alpha_1 - \beta_1$$

$$\therefore \quad \alpha_1 = 1 \text{ and } \beta_1 = 2$$

$$\Rightarrow \quad \boxed{h_1^{(0)}(\xi) = \tfrac{1}{4}(2+\xi)(1-\xi)^2}$$

Similarly $\boxed{h_2^{(0)}(\xi) = \tfrac{1}{4}(2-\xi)(1+\xi)^2}$

For the derivative basis functions

$$\left.\begin{array}{l} 0 = h_1^{(1)}(-1) = (\delta_1 - \gamma_1) \\ 1 = h_1^{(1)}(-1) = 2\gamma_1 - \delta_1 \end{array}\right\} \Rightarrow \delta_1 = \gamma_1 = 1 \Rightarrow \boxed{h_1^{(1)}(\xi) = \tfrac{1}{4}(1+\xi)(1-\xi)^2}$$

$$\text{Similarly } \boxed{h_2^{(1)}(\xi) = \tfrac{1}{4}(\xi-1)(1+\xi)^2}$$

Expression of basis functions on an arbitrary interval $[x_i, x_{i+1}]$**.**

$$\begin{aligned} \text{Use the linear transformation } x(\xi) &= x_i N_1^{(1)}(\xi) + x_{i+1} N_2^{(1)}(\xi) \\ &= x_i \frac{1}{2}(1-\xi) + x_{i+1}\frac{1}{2}(1+\xi) \\ &= \left(\frac{x_i + x_{i+1}}{2}\right) + \frac{(x_{i+1} - x_i)}{2}\xi \\ \text{The inverse transformation is: } \xi(x) &= \frac{2x - (x_i + x_{i+1})}{(x_{i+1} - x_i)} = \frac{2x - (x_i + x_{i+1})}{\Delta x_i} \end{aligned}$$

$$\begin{aligned} 1+\xi &= \frac{x_{i+1} - x_i + 2x - x_i - x_{i+1}}{x_{i+1} - x_i} = \frac{2(x - x_i)}{(x_{i+1} - x_i)} \Rightarrow 2 + \xi = \frac{\Delta x_i + 2(x - x_i)}{\Delta x_i} \\ 1 - \xi &= \frac{2(x_{i+1} - x)}{x_{i+1} - x_i} \qquad \Rightarrow (2 - \xi) = \frac{\Delta x_i + 2(x_{i+1} - x)}{\Delta x_i} \end{aligned}$$

$$\boxed{\begin{aligned} \therefore h_i^{(0)}(x) &= \frac{[\Delta x_i + 2\,(x - x_i)]\,(x_{i+1} - x)^2}{(\Delta x_i)^3} \\ h_{i+1}^{(0)}(x) &= \frac{[\Delta x_i + 2\,(x_{i+1} - x)]\,(x - x_i)^2}{(\Delta x_i)^3} \\ h_i^{(1)}(x) &= \frac{(x - x_i)(x_{i+1} - x)^2}{(\Delta x_i)^2} \\ h_{i+1}^{(1)}(x) &= -\frac{(x_{i+1} - x)(x - x_i)^2}{(\Delta x_i)^2} \end{aligned}}$$

Note: $\Delta x_i = x_{i+1} - x_i$

$$\begin{aligned} \tfrac{d\,h_i}{d\xi}(\xi) &= \tfrac{d\,h_i}{dx}(\xi(x)) \cdot \tfrac{dx}{d\xi} \\ &= \tfrac{d\,h_i}{dx}(\xi(x)) \cdot \tfrac{\Delta x_i}{2} \end{aligned}$$

Error involved:
For function values the error is given by:

$$|f(x) - h(x)| \le ||f^{(4)}||_\infty \tfrac{\Delta x^4}{384}$$

while for derivatives the error is :

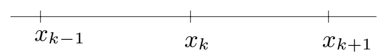
$$|f'(x) - h'(x)| \le ||f^{(4)}||_\infty \tfrac{\sqrt{3}\Delta x^3}{216}$$

Piecewise cubic spline interpolation:

NDOF: $4N - 3(N-1) = N + 1 + 2 \Rightarrow$ specify $f(x_i)$ at $x_0, \ldots, x_N$.
$+\ 2$ extra conditions

Since we have similar piecewise cubic polynomials to the PWCH polynomials on each subinterval but with additional continuity required at the $N-1$ interior nodes, our starting point is the Hermite cubic basis expansion. We then impose additional conditions to make up for the derivatives $f'(x_i)$ which are not known (or required) in the case of splines.

On $[x_k, x_{k+1}]$



$$\begin{aligned} s(x) &= f_k \frac{[\Delta x_k + 2\,(x - x_k)]\,(x_{k+1} - x)^2}{(\Delta x_k)^3} + f_{k+1} \frac{[\Delta x_k + 2\,(x_{k+1} - x)]\,(x - x_k)^2}{(\Delta x_k)^3} \\ &\quad + s'_k \frac{(x - x_k)(x_{k+1} - x)^2}{(\Delta x_k)^2} + s'_{k+1} \frac{(x - x_{k+1})(x - x_k)^2}{(\Delta x_k)^2} \end{aligned}$$

where the f'_k and f'_{k+1} from the Hermite expansion have been replaced by the unknown quantities s'_k and s'_{k+1} which are to be determined by a system of equations which ensure that $s''(x)$ is continuous at internal nodes.

The first and second derivatives of $s(x)$ must be continuous at the points x_j. Continuity of the first derivative is already obtained by our choice of basis functions. So we apply continuity of $s''(x)$ to get equations for the coefficients s'_k. That is, we calculate $s''(x)$ on the two intervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ and require continuity at x_k. After some algebra.

$$\Delta x_k s'_{k-1} \quad + \quad 2(\Delta x_k + \Delta x_{k-1}) s'_k + \Delta x_{k-1} s'_{k+1} = 3(f[x_k, x_{k+1}]\Delta x_{k-1} + f[x_{k-1}, x_k]\Delta x_k)$$

We have $N-1$ equations and $(N+1)$ unknowns $s'_0, s'_1, \ldots, s'_N$ so we need 2 more conditions. Say we specify

(I.) $f'_0 = s'_0$ and $s'_N = f'_N$ then on a uniform mesh $\Delta x_k = \Delta x$:

$$\begin{bmatrix} 4 & 1 & 0 & & & \cdots & 0 \\ 1 & 4 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & 4 & 1 & & & \\ & 0 & & & \ddots & & \vdots \\ \vdots & & & \ddots & & & \\ & & & & & & 0 \\ & & & & & & 1 \\ 0 & & \cdots & & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} s'_1 \\ s'_2 \\ \\ \vdots \\ \\ \vdots \\ \\ s'_{N-1} \end{bmatrix} = 3 \begin{bmatrix} f[x_1, x_2] + f[x_0, x_1] \\ f[x_2, x_3] + f[x_1, x_2] \\ \\ \vdots \\ \\ \vdots \\ \\ f[x_{N-1}, x_N] + f[x_{N-2}, x_{N-1}] \end{bmatrix} \begin{matrix} - \\ \\ \\ \\ \\ \\ \\ - \end{matrix} \begin{bmatrix} s'_0 \\ \\ \vdots \\ \\ \\ \vdots \\ \\ s'_N \end{bmatrix}$$

A tridiagonal system of equations – easy to solve!