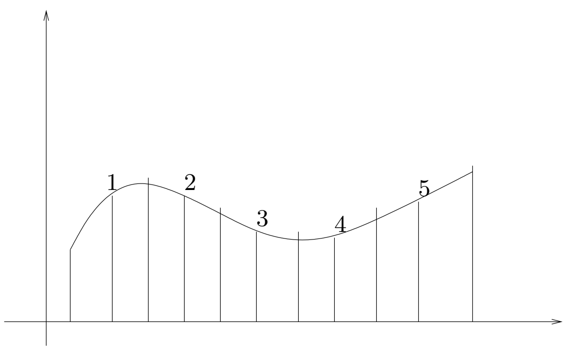
The Runge phenomenon – problems with high degree interpolants

Let $f(x) = \dfrac{1}{1+25x^2}$ and try to pass an interpolation polynomial through $n = 11$ equidistant points on the interval $[-1, 1]$.

Note the oscillations in the interpolant which renders it basically useless for interpolation as an approximation for the derivative and for the purposes of numerical integration.

Solutions to the problem of interpolating over many points.

- Smooth the wrinkles in the interpolating polynomial by *fitting* a lower degree polynomial – no longer interpolation.
- Restrict ourselves to a string of lower degree polynomials each of which are only applied over one or two subintervals—*use piecewise polynomial interpolation.*



Chebyshev interpolation (Minimax Optimization)

Question: Is it possible to choose the interpolation points $\{x_i\}_{i=0}^N$ so that the maximum absolute error (i.e. $||e_n(x)||_\infty$) is minimized?

Recall: $e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n) \qquad \xi \in (a,b)$.

For convenience we consider the interval $[-1, 1]$. There is no loss of generality in this assumption as the transformation $x = \frac{t(b-a)+(a+b)}{2}$: can be used to transform the problem $x \in [a, b]$ into one in which the independent variable is $t \in [-1, 1]$.

Important Properties of the Chebyshev Polynomials:

1. **Definition** Let $z = e^{i\theta}$ be a point on the unit circle. The associated x coordinate is $x = cos\theta$ or $\theta = \cos^{-1} x$ where $x \in [-1, 1]$. Define the nth degree Chebyshev polynomial to be $T_n(x) = \cos n\theta$. Thus $T_0(x) = Re(z^0) = cos0 = 1$, $T_1(x) = Re(z^1) = cos\theta = (z + z^{-1})/2 = x$, $T_2(x) = Re(z^2) = cos2\theta = (z^2 + z^{-2})/2 = \frac{1}{2}(z + z^{-1})^2 - 1 = 2x^2 - 1, \dots$.

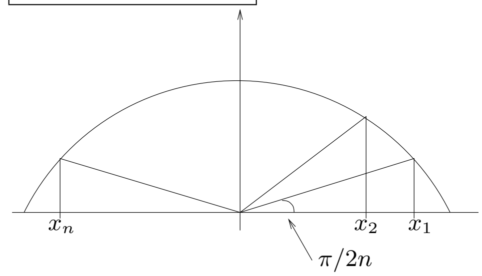
2. **Recursion**: The identity $\cos n\theta = 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta$ implies the recursion $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

   Starting with $T_0(x) = 1$ and $T_1(x) = x$ the recursion yields $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x, \dots$. Note that the leading coefficient of $T_n(x)$ is 2^{n-1}.

3. **Orthogonality**: $\int\limits_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x)T_n(x)dx = \delta_{mn}(\pi/2)$.

4. **Max/Min values and roots**:

   - **Roots**: $T_n(x) = \cos n\theta = 0$ when $n\theta = (2k+1)\frac{\pi}{2} \quad k = 0, \ldots, n-1 \Rightarrow \boxed{x_k = \cos\left[\frac{(2k+1)\pi}{2n}\right]}$



   - Max/Min: There are $n-1$ extrema between the n roots (Rollé). In addition

$$T_n(-1) = \cos n\left(\cos^{-1}(-1)\right) = \cos(n\pi) = (-1)^n$$
$$T_n(+1) = \cos n\left(\cos^{-1}(1)\right) = \cos(n2\pi) = 1$$

     $\therefore T_n(x)$ has $n+1$ extreme values on $[-1,+1]$ which are either -1 or $+1$.

In order to minimize the maximum absolute error $\max\limits_{x\in[-1,1]} |f(x) - p_n(x)|$ we must choose the $\{x_i\}$ so that $\max\limits_{x\in[-1,1]} |(x-x_0)\ldots(x-x_n)|$ is minimized since we have no control over the term $\frac{f^{(n+1)}(\xi)}{(n+1)!}$ which may be regarded as a constant for our purposes.

If we choose $x_0, x_1, \ldots, x_n$ to be the zeros of $T_{n+1}(x)$ then

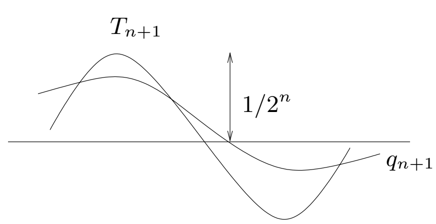
$$(x-x_0)(x-x_1)\ldots(x-x_n) = \frac{T_{n+1}(x)}{2^n} \quad : \quad x_k = \cos\left[\frac{(2k+1)\pi}{2(n+1)}\right] \quad k = 0, 1, \ldots, n.$$

Claim: $\frac{T_{n+1}(x)}{2^n}$ is the polynomial of degree $(n+1)$ that has the smallest $||\cdot||_\infty$ value over the interval $[-1,1]$.

Proof: Assume q_{n+1} is a polynomial of degree $n+1$ with leading coefficient 1 that achieves a lower $||\cdot||_\infty$ norm, i.e. $||q_{n+1}||_\infty \leq ||T_{n+1}||_\infty$.

Now $||T_{n+1}/2^n||_\infty = 1/2^n$ is achieved $n+2$ times within $[-1,1]$. By definition $|q_{n+1}(x)| < 1/2^n$ at each of the $n+2$ extreme points.

Thus $D(x) = \frac{T_{n+1}}{2^n} - q_{n+1}$ is a polynomial of degree $\leq n$ and has the same sign as T_{n+1} at each of the $n+2$ extreme points.



$\Rightarrow D(x)$ must change sign $n+1$ times on $[-1,1]$ which is impossible for a polynomial of degree $\leq n$. $\Rightarrow$ contradiction. ■

Conclusion:

1. If we choose the $\{x_k\}$ to be the Chebyschev points then $||f(x) - p_n(x)||_\infty$ is the smallest for all polynomials of degree n.

2. In the Chebyschev case the error is more uniformly distributed over the interval $[-1,1]$ than for any other polynomial.

Optimal distribution of sample points:

Consider the monic polynomial $p(z) = (z-z_1)(z-z_2)...(z-z_N) = \prod_{k=1}^{N}(z-z_k)$ and consider the absolute value of $p(z)$ which can be expressed as follows:

$$\begin{aligned} |p(z)| &= \prod_{k=1}^{N} |z-z_k| \\ &= e^{N\left[\frac{1}{N}\sum_{k=1}^{N}\log|z-z_k|\right]} \\ &= e^{N\phi_N(z)} \end{aligned}$$

where $\phi_N(z) = \frac{1}{N}\sum_{k=1}^{N}\log|z-z_k|$. Since each of the terms $\log|z-z_k|$ satisfies Laplace's equation and represents the potential due to a point charge, $\phi_N(z)$ can be interpreted as the potential due to N distributed charges of strength $\frac{1}{N}$ located at each of the points z_k. The question is: what distribution $\{z_k\}$ of charges will result in the minimum magnitude of the polynomial $p(z)$ on $[-1,1]$? We observe that if $\phi_N(z)$ is approximately constant on $[-1,1]$ then $p(z)$ will be close to constant. On the other hand if $\phi_N(z)$ is varies significantly on $[-1,1]$ then $p(z)$ can have exponentially large fluctuations on on $[-1,1]$.

Continuous distribution of points:

Introduce a function $\rho_N(x)$ defining the density of points per unit length, then

$$\phi_N(z) = \frac{1}{N}\sum_{k=1}^{N}\rho_N(x_k)\Delta x \log|z-x_k|$$

where $\int_{-1}^{1}\rho_N(x)dx = 1$. As examples we consider the uniform density $\rho_N(x) = \frac{N}{2}$ and the Chebyshev density $\rho_N(x) = \frac{N}{\pi\sqrt{1-x^2}}$.

For a large number of sample points $N \to \infty$ we obtain the following integral equation:

$$\phi(z) = \int_{-1}^{1}\rho(x)\log|z-x|dx$$

1. For a uniform density $\rho = \frac{1}{2}$ we obtain the potential:

$$\phi(z) = -1 + \frac{1}{2}Re\{(z+1)\log(z+1) - (z-1)\log(z-1)\}$$

We observe that $\phi(0) = -1$ and that $\phi(\pm 1) = -1 + \log 2$ so that

$$|p(z)| \sim \begin{cases} e^{-N} & z \to 0 \\ e^{-N}2^N & z \to \pm 1 \end{cases}$$

2. For the Chebyshev density $\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}$ the associated potential is:

$$\phi(z) \quad = \quad \log \left|\frac{z-\sqrt{z^2-1}}{2}\right|$$

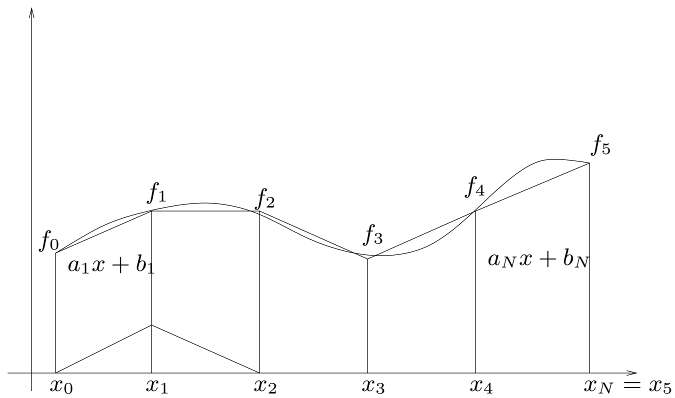
From the limiting values $\phi(0) = \log 2^{-1}$ and $\phi(\pm 1) = \log 2^{-1}$ we observe that

$$|p(z)| = e^{N\phi_N} \quad \sim \quad 2^{-N} \qquad \text{throughout } [-1, 1]$$

0.2 Piecewise polynomial interpolation

Idea: to limit the rapid oscillations of high degree polynomial interpolants by stringing together lower degree polynomial interpolants.

1. **Piecewise linear interpolation**:



Degree of freedom analysis:

$$\left.\begin{array}{ll} N \text{ intervals} & \\ a_ix + b_i & \text{2 coefficients for interval} \end{array}\right\} \Rightarrow 2N \text{ unknowns}$$

Impose continuity between interior nodes $\Rightarrow N-1$ constraints

$$2N - (N-1) = N+1 \quad \begin{array}{l}\text{degrees of freedom which can be} \\ \text{determined by specifying } f \text{ as } N+1 \text{ points } x_0, \ldots, x_n.\end{array}$$

Convenient basis function representation of the PWL interpolants of f:

Let

$$N_i^1(x) \quad = \quad \begin{cases} \left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right) & x \in [x_{i-1}, x_i] \\ \left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right) & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases}$$