Method 3.1: **Special case of equally spaced meshpoints**:
Assume

$$x_k = a + kh \quad \text{where } h = \frac{b-a}{N}, \quad k = 0, \dots, N$$

DIFFERENCE OPERATORS

Forward:

$$\begin{aligned} \Delta f_n &= f_{n+1} - f_n \\ \Delta^2 f_n &= \Delta f_{n+1} - \Delta f_n = f_{n+2} - 2f_{n+1} + f_n \end{aligned}$$

Backward:

$$\begin{aligned} \nabla f_n &= f_n - f_{n-1} \\ \nabla^2 f_n &= \nabla f_n - \nabla f_{n-1} = f_n - 2f_{n-1} + f_{n-2} \end{aligned}$$

Central:

$$\begin{aligned} \delta f_n &= f_{n+1/2} - f_{n-1/2} \\ \delta^2 f_n &= \delta f_{n+1/2} - \delta f_{n-1/2} = f_{n+1} - 2f_n + f_{n-1} \end{aligned}$$

Shift:

$$E f_n = f_{n+1}$$

Average:

$$\mu f_n = \frac{1}{2}\left\{f_{n+1/2} + f_{n-1/2}\right\}$$

Derivative:

$$D f_n = \frac{d}{dx} f|_{x=x_n}$$

Relationship between SHIFT and difference operators:

$$\Delta = (E-1) \qquad E = (1+\Delta)$$
$$\nabla = (1-E^{-1}) \qquad E^{-1} = (1-\nabla)$$

Interpolation formula:

$$\begin{aligned} f_p = E^{p-i} f_i &= (1+\Delta)^{p-i} f_i \\ &= \left\{ 1 + \frac{(p-i)}{1!}\Delta + \frac{(p-i)(p-i-1)}{2!}\Delta^2 + \ldots , \right\} f_i \end{aligned}$$

$$\boxed{f_p = \sum_{k=0}^{m} \binom{p-i}{k} \Delta^k f_i}$$

Gregory-Newton Interpolation formula

Eg: Derive the identity $s_N = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}$

f_n	Δ	Δ^2	Δ^3
$s_0 = 0$	1	1	0
$s_1 = 1$	2	1	0
$s_2 = 3$	3	1	0
$s_3 = 6$	4	1	
$s_4 = 10$			

$$\therefore \quad s_N = \left\{ 1 + N\Delta + \tfrac{N(N-1)}{2!}\Delta^2 \right\} s_0$$
$$= 0 + N \cdot 1 + \tfrac{N(N-1)}{2} \cdot 1 = \tfrac{2N+N^2-N}{2} = \tfrac{N(N+1)}{2}$$

Using backward differences:

$$f_{n-p} = E^{-p} f_n = (1-\nabla)^p f_n = \sum_{k=0}^{m} (-1)^k \binom{p}{k} \nabla^k f_n$$

$$f_{n+s} = E^{s} f_n = (1-\nabla)^{-s} f_n = \sum_{k=0}^{m} (-1)^k \binom{-s}{k} \nabla^k f_n$$

$$\text{where } \binom{y}{k} = \begin{cases} \frac{y(y-1)\ldots(y-k+1)}{k!} & k > 0 \\ 1 & k = 0. \end{cases}$$

Example 2:

$f(x) = x^3 + 2x + 1$

x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	1	3	6	6
1	4	9	12	6
2	13	21	18	
3	34	39		
4	73			

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	1	12	48	48
2	13	60	96	48
4	73	156	144	
6	229	300		
8	529			

$$
\begin{aligned}
E^p f_0 &= \left(1 + p\Delta + \frac{p^2 - p}{2}\Delta^2 + \left(\frac{p^3 - 3p^2 + 2p}{6}\right)\Delta^3\right) f_0 \\
&= 1 + p \cdot 3 + \frac{p^2 - p}{2} \cdot 6 + \frac{p^3 - 3p^2 + 2p}{6} \cdot 6 \\
&= 1 + 3p + 3p^2 - 3p + p^3 - 3p^2 + 2p \\
&= p^3 + 2p + 1
\end{aligned}
$$

Note: in general $p = \left(\frac{x - x_0}{h}\right)$ or $x = x_0 + ph$

$$
\begin{aligned}
E^p f_0 &= 1 + p \cdot 12 + \frac{(p^2 - p)}{2} \cdot 48 + \left(\frac{p^3 - 3p^2 + 2p}{6}\right) \cdot 48 = 1 + 12p + 24p^2 - 24p + 8p^3 - 24p^2 + 16p \\
&= 1 + 4p + 8p^3 = 1 + 4\left(\frac{x}{2}\right) + 8\left(\frac{x}{2}\right)^3 = 1 + 2x + x^3 \\
p = \left(\frac{x - x_0}{h}\right) &= \frac{x - 0}{2}
\end{aligned}
$$

0.1.3 Taylor Series and numerical differentiation

(Basically differentiate polynomial interpolants)

$$
\begin{aligned}
y_{n+1} &= y_n + hDy_n + h^2\frac{D^2}{2!}y_n + \ldots = \left[1 + hD + \frac{h^2D^2}{2!} + \ldots\right] y_n = e^{hD} y_n \\
\therefore E &= e^{hD} \\
hD &= \ln E = \ln(1 + \Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \ldots \\
&= -\ln(1 - \nabla) = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots \\
\delta y_n &= \left(E^{1/2} - E^{-1/2}\right) y_n \\
&= 2\left\{\frac{e^{hD/2} - e^{-hD/2}}{2}\right\} y_n = 2\sinh\left(\frac{hD}{2}\right) y_n \\
&\Rightarrow hD = 2\sinh^{-1}\left(\frac{\delta}{2}\right) = 2\left\{\left(\frac{\delta}{2}\right) - \frac{1}{2^3 3}\delta^3 + \ldots\right\} \\
\therefore hDy_n &\approx \left\{\begin{array}{ll} \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \ldots & \\ \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots & \\ \quad \mu\delta & -\frac{1}{3}\mu\delta^3 + \ldots \end{array}\right\} y_n \\
h^2D^2y_n &\approx \left\{\begin{array}{ll} \Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \ldots & \\ \nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots & \\ \quad \delta^2 & -\frac{1}{12}\delta^4 + \end{array}\right\} y_n
\end{aligned}
$$

$$h^3D^3y_n \approx \begin{cases} \Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \dots \\ \nabla^3 + \frac{3}{2}\nabla^4 + \frac{7}{4}\nabla^5 + \dots \\ \quad \mu\delta^3 \qquad\qquad\qquad -\frac{1}{4}\mu\delta^5 + \dots \end{cases}$$

$$h^4D^4y_n \approx \begin{cases} \Delta^4 - 2\Delta^5 + \frac{17}{6}\Delta^6 - \dots \\ \nabla^4 + 2\nabla^5 + \frac{17}{6}\Delta^6 + \dots \\ \quad \delta^4 \qquad\qquad\qquad -\frac{1}{6}\delta^6 + \end{cases}$$

To determine the error terms involved in finite difference approximations consider the following expansions.

$$\begin{aligned} y_{n\pm1} &= y_n \pm hy'_n + \frac{h^2}{2}y''_n + \dots \\ y_{n+2} &= y_n + 2hy'_n + 2h^2y''_n + \dots \end{aligned}$$

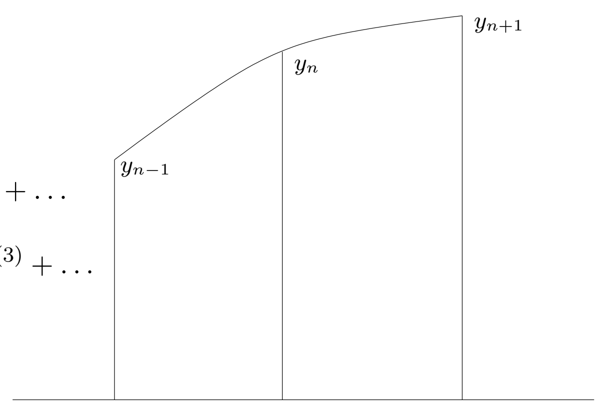
Discretization or Truncation Error

Eg. $hDy_n \approx \Delta y_n = y_{n+1} - y_n = hy'_n + \frac{h^2}{2}y''(\xi_1)$ $\quad 0(h)$ forward

$\approx \nabla y_n \quad y_n - y_{n-1} = hy'_n - \frac{h^2}{2}y''(\xi_2)$ $\quad 0(h)$backward

$\approx \mu\delta y_n = \frac{y_{n+1} - y_{n-1}}{2} = hy'_n + \frac{h^3}{6}y^{(3)}(\xi_3)$ $\quad 0(h^2)$central

Similarly:



$$\begin{aligned} h^2D^2y_n \approx \Delta^2 y_n &= y_{n+2} - 2y_{n+1} + y_n \\ &= \cancel{y_n} + 2\cancel{hy'_n} + 2h^2y''_n + \frac{8h^3}{6}y^{(3)} + \dots \\ &\quad -2\left\{\cancel{y_n} + \cancel{hy'_n} + \frac{h^2}{2}y''_n + \frac{h^3}{6}y^{(3)} + \dots\right. \\ &\quad +\cancel{y_n} \\ &= h^2y''_n + h^3y^{(3)}(\xi) \qquad 0(h) \\ \approx \delta^2 y_n &= y_{n+1} - 2y_n + y_{n-1} \\ &= h^2y''_n + \frac{h^4}{12}y^{(4)}(\xi) \qquad 0(h^2) \end{aligned}$$

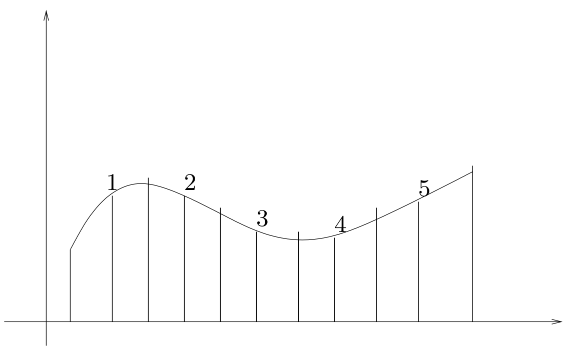
The Runge phenomenon – problems with high degree interpolants

Let $f(x) = \dfrac{1}{1+25x^2}$ and try to pass an interpolation polynomial through $n = 11$ equidistant points on the interval $[-1, 1]$.

Note the oscillations in the interpolant which renders it basically useless for interpolation as an approximation for the derivative and for the purposes of numerical integration.

Solutions to the problem of interpolating over many points.

- Smooth the wrinkles in the interpolating polynomial by *fitting* a lower degree polynomial – no longer interpolation.
- Restrict ourselves to a string of lower degree polynomials each of which are only applied over one or two subintervals—*use piecewise polynomial interpolation.*



Chebyshev interpolation (Minimax Optimization)

Question: Is it possible to choose the interpolation points $\{x_i\}_{i=0}^{N}$ so that the maximum absolute error (i.e. $||e_n(x)||_\infty$) is minimized?

Recall: $e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n) \qquad \xi \in (a,b)$.

For convenience we consider the interval $[-1, 1]$. There is no loss of generality in this assumption as the transformation $x = \frac{t(b-a)+(a+b)}{2}$: can be used to transform the problem $x \in [a, b]$ into one in which the independent variable is $t \in [-1, 1]$.

Important Properties of the Chebyshev Polynomials:

1. **Definition** Let $z = e^{i\theta}$ be a point on the unit circle. The associated x coordinate is $x = cos\theta$ or $\theta = \cos^{-1} x$ where $x \in [-1, 1]$. Define the nth degree Chebyshev polynomial to be $T_n(x) = \cos n\theta$. Thus $T_0(x) = Re(z^0) = cos0 = 1$, $T_1(x) = Re(z^1) = cos\theta = (z + z^{-1})/2 = x$, $T_2(x) = Re(z^2) = cos2\theta = (z^2 + z^{-2})/2 = \frac{1}{2}(z + z^{-1})^2 - 1 = 2x^2 - 1, \dots$.

2. **Recursion**: The identity $\cos n\theta = 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta$ implies the recursion $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

   Starting with $T_0(x) = 1$ and $T_1(x) = x$ the recursion yields $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x, \dots$. Note that the leading coefficient of $T_n(x)$ is 2^{n-1}.

3. **Orthogonality**: $\int\limits_{-1}^{1} \dfrac{1}{\sqrt{1-x^2}} T_m(x)T_n(x)dx = \delta_{mn}(\pi/2)$.