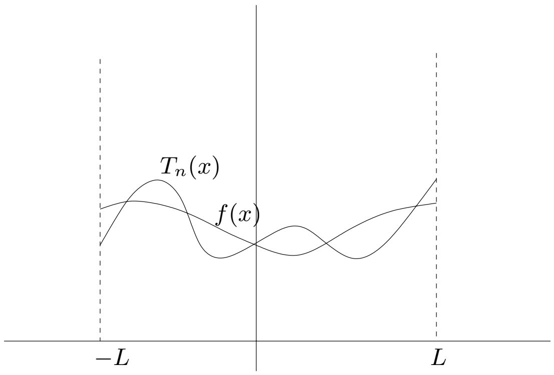
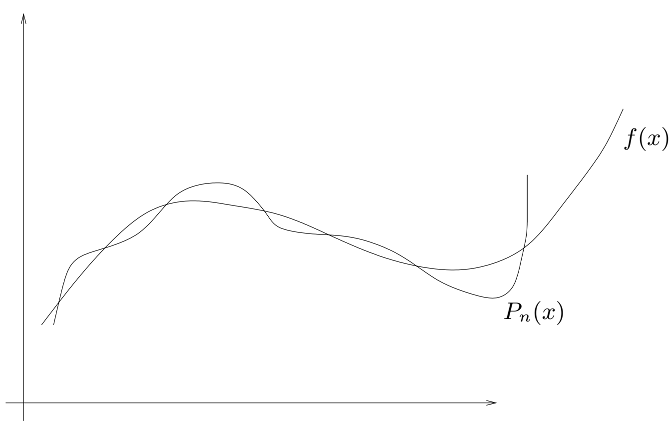


0.1.2 A systematic study of polynomial interpolation and extrapolation

- Was very important before the advent of calculators and computers when we had to interpolate between tabulated function values.
- Now it is more classical but still useful for theoretical studies of numerical approximation schemes.
- Has recently undergone a revitalization with the advent of computer graphics, image storage and reconstruction

Problem: Say we know f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$. Then how can we determine good approximations to f at intermediate points.

Idea: Approximate f by a polynomial that passes through the points and evaluate the polynomial at the desired points.



Method 1: (Directly)

$$f(x) \approx p_n(x) = a_0 + a_1 x + \ldots + a_n x^n$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & & x_0^n \\ \vdots & \vdots & & & \\ \vdots & \vdots & & & \\ 1 & x_n & x_n^2 & & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \\ f(x_n) \end{bmatrix}$$

Very difficult matrix problem to solve numerically; roughly speaking 'neighboring points give roughly the same information about the coefficients' so the matrix is nearly singular.

Method 2: Lagrange interpolation of degree n.

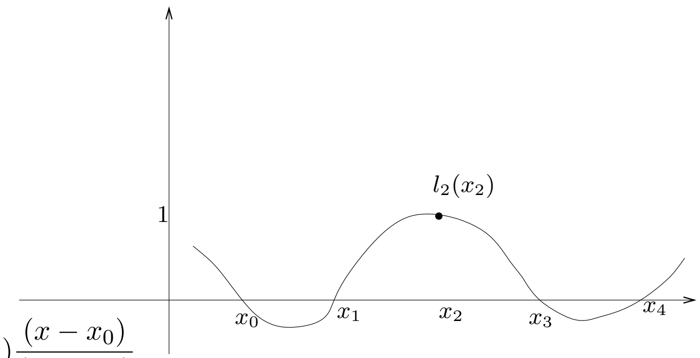
Define the following polynomial basis functions $\ell_k(x)$ each of degree n such that

$$\ell_k(x_j) = \delta_{kj} = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases}$$

$$\ell_k(x) = \frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)} = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}, \text{ note } \sum_{k=0}^{N} \ell_k(x) = 1.$$

Then we have the following representation

$$f(x) \approx \sum_{k=0}^{n} f(x_k)\ell_k(x) = p_n(x).$$



Eg.

$$\begin{aligned} n=1: \quad f(x) &\approx f(x_0)\frac{(x-x_1)}{(x_0-x_1)} + f(x_1)\frac{(x-x_0)}{(x_1-x_0)} \\ &= f(x_0)\frac{[(x-x_0)+(x_0-x_1)]}{(x_0-x_1)} + f(x_1)\frac{(x-x_0)}{(x_1-x_0)} \\ &= f(x_0) + \frac{f(x_1)-f(x_0)}{(x_1-x_0)}(x-x_0) \qquad \text{linear interpolation} \end{aligned}$$

Problem with Lagrange:

- it is expensive to evaluate the $\ell_k(x)$; requires $\{2(n+1)\}$ multiplications/divisions and $(2n+1)$ additions after function values have been corrected for denominators}
  whereas a polynomial in a power form can be evaluated in n multiplications and n additions

$$\begin{aligned} p_n(x) &= a_0 + a_1 x + \dots + a_n x^n \\ &= \quad \underbrace{+(a_{n-2} + (\underbrace{a_{n-1} + a_n x}_{p_1})x}_{p_2} \end{aligned}$$

$$\begin{aligned} p_0 &= a_n & & \text{Horner's Rule} \\ \text{for} \quad k &= 1,\dots,n & n* \text{ and } n+ & \\ p_k &= a_{n-k} + p_{k-1}x & & \end{aligned}$$

- If we wanted a higher degree interpolant, then we would have to throw all the previous information away and recalculate a new interpolant.

Method 3: Newton's divided difference table:
Lagrange method for $n = 2$

$$\begin{aligned}
f(x) &\approx f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= f(x_0)\left\{\frac{[(x-x_0)+x_0-x_1]\,[(x-x_0)+(x_0-x_2)]}{(x_0-x_1)(x_0-x_2)}\right\} + f(x_1)\frac{(x-x_0)\,[(x-x_1)+(x_1-x_2)]}{(x_1-x_0)(x_1-x_2)} \\
&\qquad + f(x_2)\frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)} \\
&= f(x_0) + f(x_0)\frac{(x-x_0)}{(x_0-x_1)} + f(x_0)\frac{(x-x_0)(x-x_1)}{(x_0-x_1)(x_0-x_2)} \\
&\qquad + f(x_1)\frac{(x-x_0)}{(x_1-x_0)} + f(x_1)\frac{(x-x_0)(x-x_1)}{(x_1-x_0)(x_1-x_2)} \\
&\qquad + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \\
\text{where } f[x_0,x_1] &= \frac{f(x_0)}{(x_0-x_1)} + \frac{f(x_1)}{(x_1-x_0)} = \frac{f(x_1)-f(x_0)}{(x_1-x_0)} \approx f'(x_0) \\
\text{and } f[x_0,x_1,x_2] &= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} \\
&= \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} \approx \frac{f''(x_0)}{2}.
\end{aligned}$$

Newton's divided difference interpolation formula:

In general: $p_n(x) = \sum_{i=0}^{n} f[x_0,x_1,\ldots,x_i]\prod_{j=0}^{i-1}(x-x_j)$, where $f[x_0,\ldots,x_i] = \frac{f[x_1,\ldots,x_i]-f[x_0,\ldots,x_{i-1}]}{x_i-x_0} = \sum_{j=0}^{i}\frac{f(x_j)}{\prod_{j\neq k}(x_j-x_k)}$

Eg.

i	x_i	$f(x_i)$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$	$f[\,,\,,\,,\,,\,]$	
0	0	-5	6				Divided
1	1	1	12	2	1	0	Difference
2	3	25	30	6	1	0	Table
3	4	55	63	11	1		
4	6	181	108	15			
5	7	289					

$$\begin{aligned}
p_3(x) &= -5 + 6(x-0) + 2(x-0)(x-1) + (x-0)(x-1)(x-3) \\
&= -5 + 6x + 2x^2 - 2x + x^3 - 4x^2 + 3x = x^3 - 2x^2 + 7x - 5
\end{aligned}$$

Note: $\frac{d}{dx}f[x_0,\ldots,x_k,x] = \lim_{h\to 0}\frac{f[x_0,\ldots,x_k,x+h]-f[x_0,\ldots,x_k,x]}{h} = \lim_{h\to 0} f[x_0,\ldots,x_k,x,x+h] = f[x_0,\ldots,x_k,x,x]$

Error Estimate: What is the error involved when we try to approximate $f(x)$ by a polynomial of degree N?

Theorem: If $f \in C^{N+1}[a,b]$ then

$$f(x) = p_N(x) + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)(x-x_1)\ldots(x-x_N) \qquad \xi \in (x_0,x_N).$$

Lemma 1: Divided difference expression for the error.

$$e_N(\bar{x}) = f(\bar{x}) - p_N(\bar{x}) = f[x_0, x_1, \ldots, x_N, \bar{x}] \prod_{j=0}^{N} (\bar{x} - x_j)(*) \text{ for any } \bar{x} \in [x_0, x_N].$$

Proof: If $\bar{x} = x_j$ then the formula $(*)$ holds.
If $\bar{x} \neq x_j \quad j = 0, \ldots, N$, then consider the polynomial $p_{N+1}(x)$ that passes through $f(x_0), \ldots, f(x_N)$ and $f(\bar{x})$. Then

$$f(\bar{x}) = p_{N+1}(\bar{x}) = p_N(\bar{x}) + f[x_0, \ldots, x_N, \bar{x}] \prod_{j=0}^{N} (\bar{x} - x_j)$$

$$\therefore \quad e_N(\bar{x}) = f(\bar{x}) - p_N(\bar{x}) = f[x_0, \ldots, x_N, \bar{x}] \prod_{j=0}^{N} (\bar{x} - x_j)$$

■

Lemma 2: (Like the Mean Value Theorem)
If f is continuous on $[x_0, x_k]$ and k times differentiable on (x_0, x_k) then there exists a $\xi \in (x_0, x_k)$ such that

$$f[x_0, x_1, \ldots, x_k] = f^{(k)}(\xi)/k!$$

Proof: $e_N(x) = f(x) - p_N(x)$ has $N + 1$ roots in $[x_0, x_N]$, namely $x_0, x_1, \ldots, x_N$.

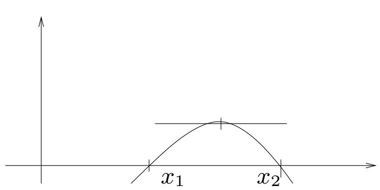
$$\begin{aligned}
\text{Rolle} \quad &\Rightarrow \quad e_n' \text{ has } N \text{ roots } \Rightarrow \ldots \Rightarrow e_N^{(N)} \text{ has 1 root in } (x_0, x_N) \\
\therefore \exists\, \xi \quad &\in \quad (x_0, x_N) \text{ such that } e_N^{(N)}(\xi) = f^{(N)}(\xi) - f[x_0, x_1, \ldots, x_N]n! = 0 \\
&\exists\, \xi \in (x_0, x_N) : f[x_0, x_1, \ldots, x_N] = f^{(N)}(\xi)/N!
\end{aligned}$$

■

Proof of Theorem:

By Lemma 1: $e_N(x) = f[x_0, \ldots, x_N, x] \prod_{j=0}^{N} (x - x_j)$.

By Lemma 2: $\exists\, \xi \in (x_0, x_N) : e_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{j=0}^{N} (x - x_j)$. ■

Recall Rolle: 

g CONT ON $[x_1, x_2]$
g DIFF ON (x_1, x_2)
$\exists\, \xi \in (x_1, x_2) : g'(\xi) = 0$