Interpolation and approximation

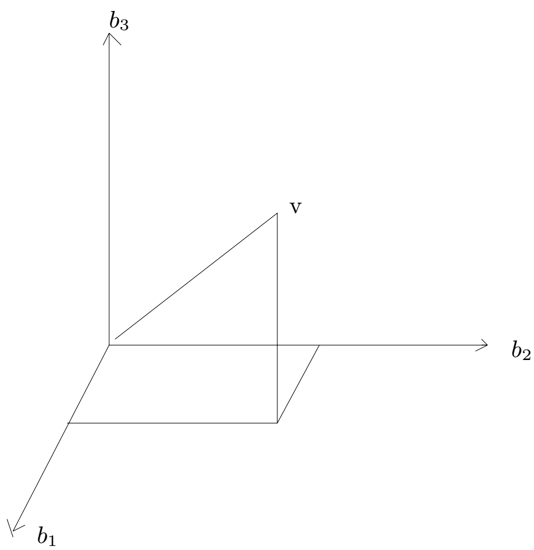
0.1 Interpolation and approximation

0.1.1 Approximating functions

A geometric view of representing arbitrary functions in terms of some basis functions:

Vector Algebra:

Representation:



$$\mathbf{v} = \sum_{i=1}^{3} \alpha_i \mathbf{b}_i \qquad (\cdot,\cdot) = \begin{array}{l}\text{dot product}\\ \text{or inner product}\end{array}$$

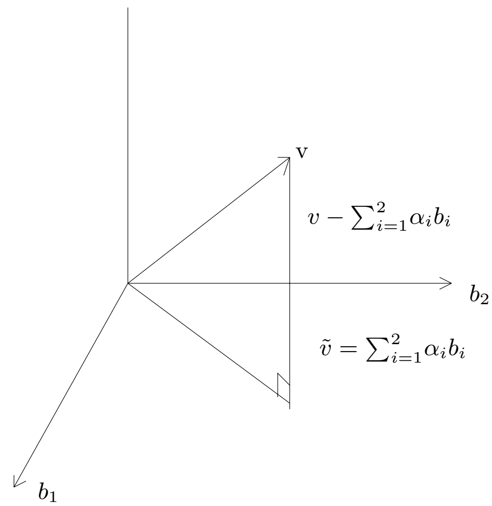
$$(\mathbf{v}, \mathbf{b}_k) = \sum_{i=1}^{3} \alpha_i (\mathbf{b}_i, \mathbf{b}_k)$$

$$\begin{bmatrix} (b_1,b_1)(b_1,b_2)(b_1,b_3) \\ (b_1,b_2)(b_2,b_2)(b_2,b_3) \\ (b_1,b_3)(b_2,b_3)(b_3,b_3) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} (\mathbf{v},b_1) \\ (\mathbf{v},b_2) \\ (\mathbf{v},b_3) \end{bmatrix}$$

$b_i \perp b_j \Rightarrow \alpha_k = (\mathbf{v}, b_k)/(b_k, b_k)$.

Can represent $\mathbf{v}$ exactly in terms of $\mathbf{b}_k$.

Approximation:



- Cannot represent $\mathbf{v}$ precisely in terms of $\mathbf{b}_1, \mathbf{b}_2$ since $\mathbf{v} \notin \text{span}\{\mathbf{b}_1, \mathbf{b}_2\}$.
- In this case we try to find the vector $\tilde{\mathbf{v}} \in \text{span}\{\mathbf{b}_1, \mathbf{b}_2\}$ which is 'closest' to $\mathbf{v}$.

Let $\tilde{\mathbf{v}} = \sum_{i=1}^{2} \alpha_i b_i$.
Then we choose α_i so that $\tilde{\mathbf{v}}$ is as close to $\mathbf{v}$ in some sense.

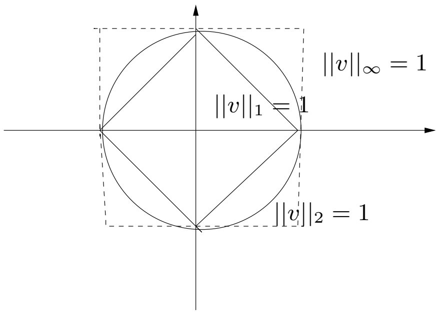
Example 1:

$$\begin{aligned} \text{minimize}||\mathbf{v} - \tilde{\mathbf{v}}||_2^2 &= (\mathbf{v} - \sum \alpha_i \mathbf{b}_i) \cdot (\mathbf{v} - \sum \alpha_j \mathbf{b}_j) \\ &= ||\mathbf{v}||^2 - 2\sum \alpha_i (\mathbf{b}_i, \mathbf{v}) + \sum_i \sum_j \alpha_i \alpha_j (\mathbf{b}_i, \mathbf{b}_j) \end{aligned}$$

$$\frac{\partial E}{\partial \alpha_k} = -2(\mathbf{b}_k, \mathbf{v}) + 2\sum_{i=1}^{2} \alpha_i (\mathbf{b}_i, \mathbf{b}_k) = 0 \qquad k = 1, 2.$$

$$\Rightarrow \left(\mathbf{v} - \sum_{i=1}^{2} \alpha_i \mathbf{b}_i, \mathbf{b}_k \right) = 0 \qquad \text{orthogonal projection}$$

A 'circle' in the different measures of distance



Example 2:

$$\min ||\mathbf{v} - \tilde{\mathbf{v}}||_1 = \min \left\{ \left| v_1 - \sum_{j=1}^{2} \alpha_j b_{j1} \right| + \left| v_2 - \sum_{j=1}^{2} \alpha_j b_{j2} \right| \right\}$$

Example 3:

$$\min ||\mathbf{v} - \tilde{\mathbf{v}}||_\infty = \min \left[\max \left\{ \left| v_1 - \sum_{j=1}^{2} \alpha_j b_{j1} \right|, \left| v_2 - \sum_{j=1}^{2} \alpha_j b_{j2} \right| \right\} \right]$$

- A function is like an ∞ dimensional vector—to describe a function numerically you would like to specify all its values—which requires an infinite number of points.

  Examples of *representations* of functions in terms of basis functions:

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{basis vectors are } \left\{\cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}. \\
&= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \ldots \quad \text{basis vectors are } \left\{1, x, x^2, \ldots\right\}. \\
&= \sum_{n=0}^{\infty} \alpha_n \phi_n(x) \quad \text{basis vectors are } \{\phi_n\}. \\
\text{Recall } a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx = \frac{\left(f, \cos\left(\frac{n\pi x}{L}\right)\right)}{\left(\cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right)}
\end{aligned}$$

  so the integral acts like an inner-product for functions—not surprising in view of the definition of the Riemann integral

$$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx \approx \frac{2L}{N}\sum_{m=0}^{N} f(x_m)\cos\left(\frac{n\pi x_m}{L}\right).$$

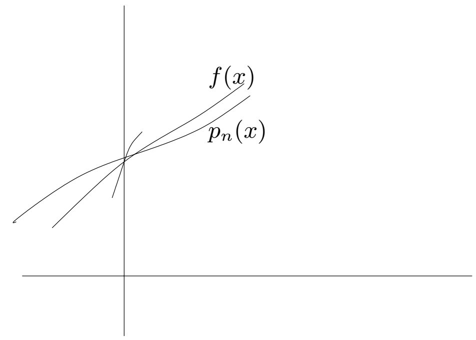
Examples of approximate representation of functions:

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=-N}^{N} c_n e^{i\frac{m\pi x}{L}} \quad \text{APPROX BY TRIG POLYNOMIALS} \\
&= a_0 + a_1 x + \ldots + a_n x^n = p_n(x) \quad \text{Polynomial approximation.} \\
&= \sum_{n=0}^{N} a_n \phi_n(x).
\end{aligned}$$

The a_k's are determined by imposing different criteria of 'closeness' between the function f and the approximation.

Examples: (1) Say we know f and n of its derivatives at a single point $f(0), f'(0), \ldots, f^{(n)}(0)$ then

$a_n = \frac{f^{(n)}(0)}{n!}$ yields the Taylor polynomial

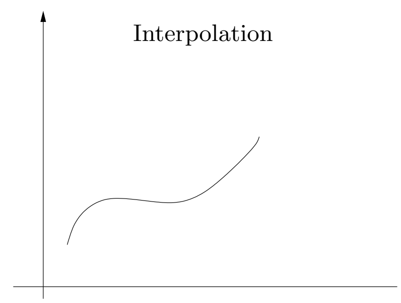


(2) Say we know f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ then

$f(x_k) = a_0 + a_1 x_k + \ldots + a_n x_k^n \qquad k = 0, \ldots, n$

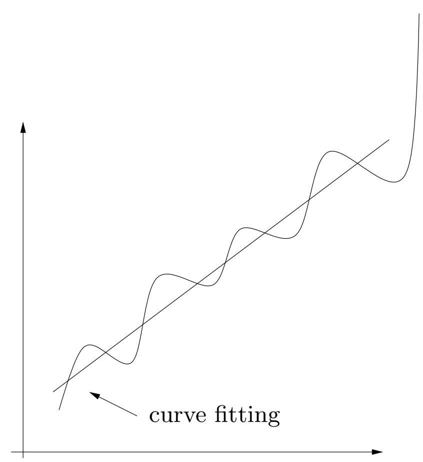
or

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ \vdots & \vdots & & & \\ \vdots & \vdots & & & \\ 1 & x_n & x_n^2 & & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ \vdots \\ \vdots \\ f(x_n) \end{bmatrix}$$



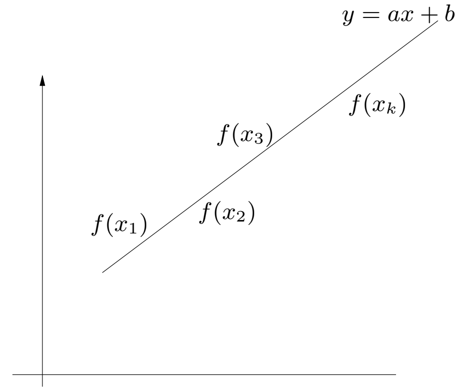
The van der Monde matrix – notoriously difficult to solve numerically.

(3) Say we know f at $m \gg n$ points $x_1, \ldots, x_n$ where some of the $f(x_k)$ may be noisy so we do not wish to place too much weight on individual points—then we perform a *least squares fit*.



Eg. 1 **Fitting a straight line — linear regression**

Fit $p_n(x) = ax + b$ to the points $(x_k, f(x_k)) \qquad k = 1, \ldots, m.$



$$\begin{aligned}
E(a,b) &= \sum_{k=1}^{m} [f(x_k) - (ax_k + b)]^2 = ||f - (ax+b)||_2^2 \\
\frac{\partial E}{\partial a} &= 2\sum_{k=1}^{m} [f(x_k) - (ax_k+b)]\,(-x_k) = 0 \\
\frac{\partial E}{\partial b} &= 2\sum_{k=1}^{m} [f(x) - (ax_k+b)]\,(-1) = 0 \\
\therefore & \begin{bmatrix} S_2 & S_1 \\ S_1 & S_0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} M_1 \\ M_0 \end{bmatrix}
\end{aligned}$$

where $\quad S_r = \sum_{k=1}^{m} x_k^r, \quad M_r = \sum_{k=1}^{m} f(x_k)x_k^r.$

Eg. 2 Fitting a trig poly of degree N to a function f on an interval $[-L, L]$. Let $f(x) \sim T_N(x) = \sum_{n=-N}^{N} c_n e^{i\left(\frac{n\pi x}{L}\right)}$ and choose c_n so that T_n is as close as possible in the least square sense.

$$\begin{aligned}
E(c_n) &= \int_{-L}^{L} \left[f(x) - \sum_{r=-N}^{N} c_r\, e^{i\left(\frac{r\pi x}{L}\right)}\right] \left[f(x) - \sum_{n=-N}^{N} c_n\, e^{i\left(\frac{n\pi x}{L}\right)}\right]^* dx \\
0 = \frac{\partial E}{\partial c_n} &= \int_{-L}^{L} e^{i\frac{n\pi x}{L}} \left[f(x) - \sum_{n=-N}^{N} c_n^*\, e^{-i\left(\frac{n\pi x}{L}\right)}\right] dx \\
\therefore 0 &= \int_{-L}^{L} e^{i\left(\frac{r\pi x}{L}\right)} f(x)dx - \sum_{n=-N}^{N} c_n^* \int_{-L}^{L} e^{+i\frac{\pi x}{L}(r-n)}\, dx
\end{aligned}$$

$$\text{but} \quad \int_{-L}^{L} e^{\frac{i\pi x}{L}(r-n)}\, dx = \left| \frac{L e^{\frac{i\pi x}{L}(r-n)}}{i\pi(r-n)} \right|_{-L}^{L} = \frac{L}{ir(r-n)} \left[e^{i\pi(r-n)} - e^{-ir(r-n)}\right] = 0, \qquad r \neq n.$$

$$= 2L \qquad\qquad r = n$$

$$\therefore c_n = \frac{1}{2L} \int_{-L}^{L} e^{-i\left(\frac{r\pi x}{L}\right)} f(x)\, dx \quad \text{which is just the Fourier coefficient}$$

$\therefore$ the Fourier coefficient does the best job in the L_2 norm