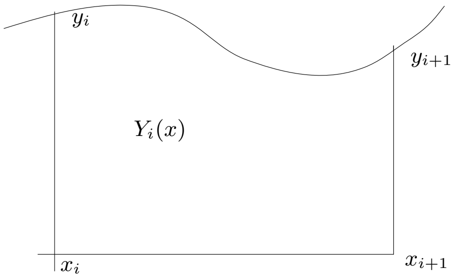
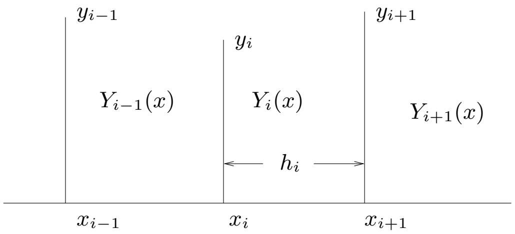
Alternative cubic spline derivation for derivative primary variables:



$$
\begin{aligned}
\text{Let } Y_i(x) &= a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \\
Y_i'(x) &= 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i \\
Y_i''(x) &= 6a_i(x-x_i) + 2b_i \\
y_i &= Y_i(x_i) = d_i \qquad y_{i+1} = Y_i(x_{i+1}) = a_i h_i^3 + b_i h_i^2 + c_i h_i + y_i \text{ where } h_i = x_{i+1} - x_i.
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } s_i' &= Y_i'(x_i) = c_i \qquad s_{i+1}' = Y_i'(x_{i+1}) = 3a_i h_i^2 + 2b_i h_i + s_i' \\
Y_i''(x_i) &= 2b_i = Y_{i-1}''(x_i) = 6a_{i-1}h_{i-1} + 2b_{i-1} \\
& \begin{bmatrix} h_i^3 & h_i^2 \\ 3h_i^2 & 2h_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} y_{i+1} - y_i - s_i' h_i \\ s_{i+1}' - s_i' \end{bmatrix} \\
a_i &= 2h_i(y_{i+1} - y_i - s_i' h_i) \Big/ (-h_i^4) - (s_{i+1}' - s_i')h_i^2 \Big/ (-h_i^4) \\
b_i &= \left\{ h_i^3(s_{i+1}' - s_i') - 3h_i^2(y_{i+1} - y_i - s_i' h_i) \right\} \Big/ (-h_i^4) \\
\therefore a_i &= -2\left(\frac{y_{i+1} - y_i}{h_i^3}\right) + \frac{s_{i+1}' + s_i'}{h_i^2} \\
b_i &= 3\left(\frac{y_{i+1} - y_i}{h_i^2}\right) - \left(\frac{s_{i+1}' + 2s_i'}{h_i}\right) \\
\therefore \frac{6}{h_i}\frac{\Delta y_i}{h_i} &- \frac{2}{h_i}\left(s_{i+1}' + 2s_i'\right) = \frac{-12}{h_{i-1}}\frac{\Delta y_{i-1}}{h_{i-1}} + 6\left(\frac{s_i' + s_{i-1}'}{h_{i-1}}\right) + \frac{6}{h_{i-1}}\frac{\Delta y_{i-1}}{h_{i-1}} - 2\frac{(s_i' + 2s_{i-1}')}{h_{i-1}} \\
\therefore 6\left(\frac{\Delta y_i}{h_i}h_{i-1}\right. &+ \left.\frac{\Delta y_{i-1}}{h_{i-1}}h_i\right) = 2h_{i-1}s_{i+1}' + 4h_{i-1}s_i' + 4h_i s_i' + 2h_i s_{i-1}' \\
\therefore & \quad \boxed{h_{i-1}s_{i+1}' + 2(h_{i-1} + h_i)s_i' + h_i s_{i-1}' = 3(f[x_{i-1}, x_i]h_i + f[x_i, x_{i+1}]h_{i-1}).}
\end{aligned}
$$

Another spline formulation with s''_i as primary variables.



$$\begin{aligned}
\text{Let } Y_i(x) &= a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \\
Y'_i(x) &= 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i \\
Y''_i(x) &= 6a_i(x-x_i) + 2b_i \\
\textbf{at } x_i: \quad y_i &= Y_i(x_i) = d_i \quad \leftarrow \text{ interpolates the function at } x_i \qquad (1) \\
y_{i+1} &= Y_i(x_{i+1}) = a_i h_i^3 + b_i h_i^2 + c_i h_i + y_i \leftarrow \text{ enforce continuity} \qquad (2)
\end{aligned}$$

$$\text{Introduce primary variable}: s''_i = Y''_i(x_i) = 2b_i \Rightarrow \boxed{b_i = s''_i/2} \qquad (3)$$

$$\text{Build in continuity of } Y'' \Rightarrow s''_{i+1} = Y''_i(x_{i+1}) = 6a_i h_i + s''_i \Rightarrow \boxed{a_i = \frac{s''_{i+1}-s''_i}{6h_i}} \qquad (4)$$

$$\begin{aligned}
\therefore Y_i(x) &= \left(\frac{s''_{i+1}-s''_i}{6h_i}\right)(x-x_i)^3 + \frac{s''}{2}(x-x_i)^2 + c_i(x-x_i) + y_i \\
(2) \Rightarrow \quad y_{i+1} &= \left(\frac{s''_{i+1}-s''_i}{6h_i}\right)h_i^3 + \frac{s''_i}{2}h_i^2 + c_i h_i + y_i \\
c_i &= \left(\frac{y_{i+1}-y_i}{h_i}\right) - \left(\frac{s''_{i+1}-s''_i}{6}\right)h_i - \frac{s''_i}{2}hi \\
&= \frac{\Delta y_i}{h_i} - \left(\frac{s''_{i+1}+2s''_i}{6}\right)h_i
\end{aligned}$$

Impose continuity of first derivatives:

$$\begin{aligned}
Y'_i(x_i) &= 3a_i(x_i-x_i)^2 + 2b_i(x_i-x_i) + c_i = c_i \\
Y'_{i-1}(x_i) &= 3a_{i-1}h_{i-1}^2 + 2b_{i-1}h_{i-1} + c_{i-1} \\
\therefore c_i = \frac{\Delta y_i}{h_i} - \left(\frac{s''_{i+1}+2s''_i}{6}\right)h_i &= 3\left(\frac{s''_i-s''_{i-1}}{6h_{i-1}}\right)h_{i-1}^2 + s''_{i-1}h_{i-1} + \left\{\frac{\Delta y_{i-1}}{h_{i-1}} - \left(\frac{s''_i+2s''_{i-1}}{6}\right)h_{i-1}\right\}
\end{aligned}$$

$$\boxed{h_{i-1}s''_{i-1}(-3+6-2) + (2h_i+2h_{i-1})s''_i + h_i s''_{i+1} \qquad = 6\left(\frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}\right)}$$

or

$$\boxed{h_{i-1}s''_{i-1} + (2h_i+2h_{i-1})s''_i + h_i s''_{i+1} \qquad = 6\left(\frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}\right)}$$

$i = 1, \ldots N-1 \qquad N-1$ EQ

2 EXTRA CONDITIONS