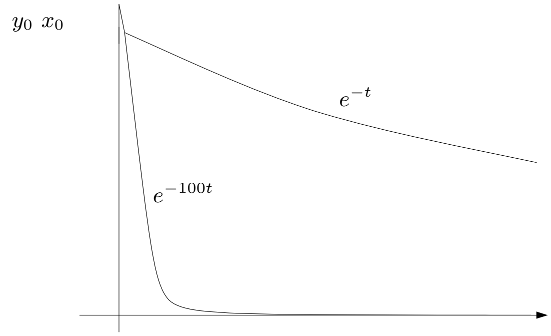
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -100 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{array}{l} x = e^{-t}x_0 \\ y = e^{-100t}y_0 \end{array}$$



• **FE**: If we were to use the FE method in the useful regime we would requre $-2 < h\lambda_k < 0$

$$\begin{aligned} \lambda_1 = -1 \quad &\Rightarrow \quad h < 2 \\ \lambda_2 = -100 \quad &\Rightarrow \quad h < 1/50 \end{aligned}$$

We do not particularly care about y since it decays to zero very rapidly but we are more interested in x which persists much longer. But to compute the system stably we woud need very small time-steps – bad news; it will take forever.

• What about using the **Trapezoidal Rule** so we don't have to worry about the timestep?

Say $Re(\lambda) \to -\infty$ and let us look at $G(z) = \frac{1+z/2}{1-z/2}$ in the case $Re(\lambda) \to -\infty$.

Let $z = \alpha + i\beta$ and let β be fixed.

$$|G(z)| = \frac{|1+z/2|}{|1-z/2|} = \frac{\sqrt{(1+\alpha/2)^2 + (\beta/2)^2}}{\sqrt{(1-\alpha/2)^2 + (\beta/2)^2}} \quad \xrightarrow{\alpha \to -\infty} \quad 1$$

solution will oscillate but will not decay!

But e^z, which $G(z)$ is supposed to approximate, is such that $e^z \to 0$ as $Re(z) \to -\infty$.

L-Stability: A numerical method for which $G(z) \to 0$ as $Re(z) \to -\infty$ is said to be L-stable or has strong decay.

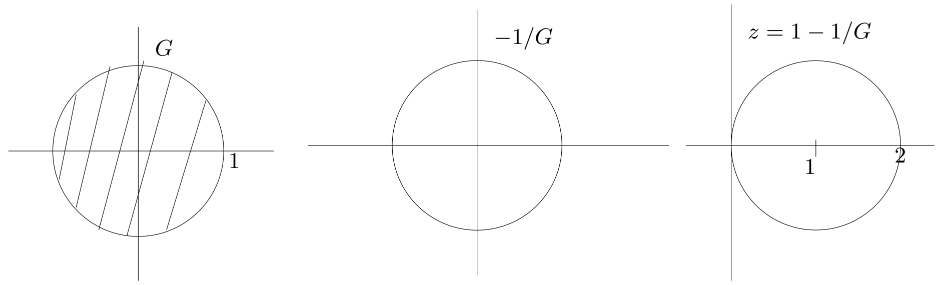
•**Example of an L-stable method**: – the Backward Euler Scheme: (BE)

$$Y_{n+1} = Y_n + hf(x_{n+1}, Y_{n+1})$$

For model problem:

$$\begin{aligned} Y_{n+1} &= Y_n + h\lambda Y_{n+1} \\ Y_{n+1} &= \frac{1}{(1-h\lambda)} Y_n = G(h\lambda)Y_n \\ G(z) = \frac{1}{1-z} & \qquad z = 1 - \frac{1}{G} \\ G(z) \to 0 \quad \text{as} & \quad Re(z) \to -\infty \end{aligned}$$

so BE is L-stable.


G
1
$-1/G$
$z = 1 - 1/G$
1
2

3.2 **Runge-Kutta methods**: – Multistage one step methods

- Can use more than 1 function evaluation (i.e., of f) per timestep.
- Idea: Use weighted average for gradients over the interval $[x_k, x_{k+1}]$ to achieve a greater accuracy when stepping from x_k to x_{k+1}.

RK2: Runge-Kutta method for order 2
Assume

$$\begin{aligned} y_{k+1} &= y_k + ahf(x_k, y_k) + bhf(x_k + \alpha h, y_k + \beta h f(x_k, y_k)) \\ &= y_k + (a+b)hf(x_k, y_k) + bh^2(\alpha f_x + \beta f\, f_y)\Big|_k \\ &\quad + bh^3\left(\frac{\alpha^2}{2} f_{xx} + \alpha\beta f\, f_{xy} + \frac{\beta^2}{2} f^2 f_{yy}\right) + O(h^4) \end{aligned}$$

Now TS:

$$y_{k+1} = y_k + hf_k + \frac{h^2}{2}\left(f_x + f\, f_y\right)\Big|_k + \frac{h^3}{3!}\left(f_{xx} + 2f\, f_{xy} + f_x f_y + f\, f_y^2 + f^2 f_{yy}\right) + \dots$$

To agree with TS up to order 2 we have

$$\boxed{a + b = 1,\ \alpha b = 1/2 = \beta b} \qquad T_n(h) = O(h^2)$$

(I). $a = 1/2 \Rightarrow b = 1/2 \Rightarrow \alpha = \beta = 1$ which is just **the improved Euler method**.

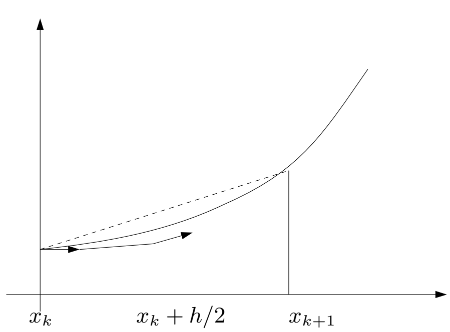
Convenient form:

$$\boxed{\begin{aligned} m_1 &= f(x_k, y_k) \qquad m_2 = f(x_{k+1}, y_k + hm_1) \\ y_{k+1} &= y_k + \frac{h}{2}(m_1 + m_2). \end{aligned}}$$

(II).

$$a = 0 \quad \rightarrow \quad b = 1, \quad \alpha = 1/2 = \beta \qquad \textbf{The Modified Euler Method.}$$

$$y_{k+1} = y_k + hf(x_k + \frac{1}{2}h, \quad y_k + \frac{1}{2}hf(x_k, y_k))$$



$$\boxed{\begin{aligned} &\text{Convenient form} \\ m_1 &= f(x_k, y_k) \\ m_2 &= f(x_k + \frac{1}{2}h,\ y_k + \frac{h}{2}m_1) \\ y_{k+1} &= y_k + hm_2 \end{aligned}}$$

RK3: $Y_{k+1} = Y_k + \frac{h}{6}(m_1 + 4m_3 + m_2) + O(h^4)$

$$\begin{aligned} m_1 &= f(x_k, Y_k) \\ m_2 &= f(x_k + h, Y_k + hm_1) \\ m_3 &= f(x_k + \frac{h}{2}, Y_k + \frac{h}{4}(m_1 + m_2)) \end{aligned}$$

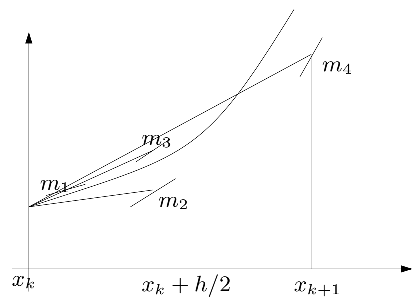
Demonstration that the method is $O(h^3)$ using the model problem. Consider $y' = \lambda y$.

$$\begin{aligned} y_{k+1} &= y_k + \frac{h}{6}\left\{\lambda y_k + 4\lambda\left[y_k + \frac{h}{4}\left(\lambda y_k + \lambda\left(y_k + h\lambda y_k\right)\right)\right] + \lambda\left(y_k + h\lambda y_k\right)\right\} \\ &= y_k + \frac{h}{6}\left\{\left(\lambda y_k + 4\lambda y_k + \lambda y_k\right) + 3h^2\lambda^2 y_k + \left(h\lambda\right)^3 y_k\right\} \\ &= \left[1 + (h\lambda) + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6}\right] y_k. \end{aligned}$$

RK4: $y_{k+1} = y_k + \frac{h}{6}\left[m_1 + 2m_2 + 2m_3 + m_4\right] + O(h^5)$

where

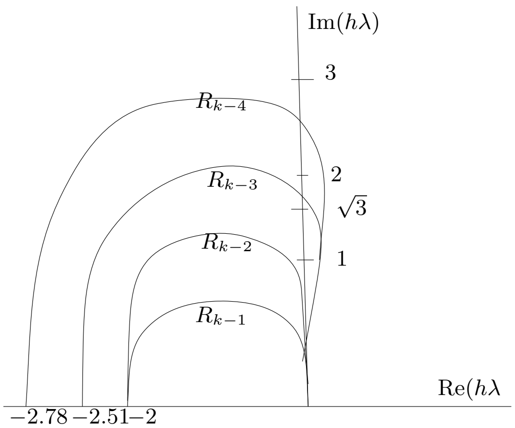
$$\begin{aligned} m_1 &= f(x_k, y_k) \\ m_2 &= f\left(x_k + \frac{h}{2}, y_k + \frac{m_1}{2}h\right) \\ m_3 &= f\left(x_k + \frac{h}{2}, y_k + \frac{m_2}{2}h\right) \\ m_4 &= f\left(x_k + h, y_k + hm_3\right) \end{aligned}$$



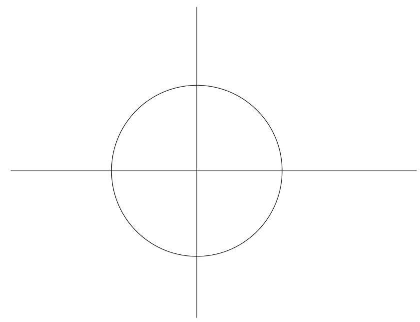
agrees with TS up to $O(h^4)$

$T_n(h) = O(h^4)$

Stability region for RK methods:



$$
\begin{aligned}
\lambda &= |\lambda| e^{i\phi} \\
y_{k+1} &= \left[1 + h\lambda + \frac{(h\lambda)^2}{2} + \ldots + \frac{(h\lambda)^P}{P!} \right] y_k \\
\theta &= 1 + (h\lambda) + \ldots + \frac{(h\lambda)^P}{P!}
\end{aligned}
$$



$$
\begin{aligned}
\theta &= 1 + re^{i\phi} + \frac{r^2}{2} e^{i2\phi} + \ldots + \frac{r^P}{P!} e^{iP\phi} \\
r &= h|\lambda|
\end{aligned}
$$

Note: For RK the order $\neq$ # of function evaluations.

# of Function Evaluations	Order of Method
2	2
3	3
4	4
5	4
6	5
7	6
$n > 8$	$n - 2$