Stability Region of the Improved Euler – RK2:

Consider the model problem $y' = \lambda y$, $y(0) = 1$, $y = e^{\lambda x}$.

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{h}{2}\left[\lambda Y_n + \lambda \left\{Y_n + h\lambda Y_n\right\}\right] \\ Y_{n+1} &= \left[1 + h\lambda + \frac{(h\lambda)^2}{2}\right] Y_n \approx e^{h\lambda} Y_n \end{aligned}$$

(Note: With Taylor Series up to $O((h\lambda)^n)$ – Be careful if you try to infer the error by looking at $G(z) = 1 + z + z^2/2$ since we would be ignoring the time stepping part.)

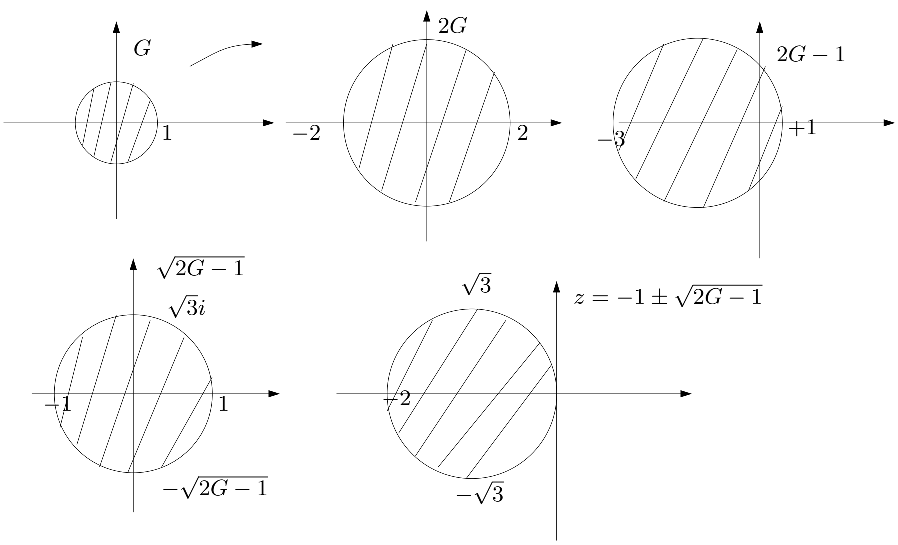
The growth factor is $G(\lambda h) = 1 + h\lambda + \frac{(h\lambda)^2}{2}$

For stability we require $|G(\lambda h)| < 1$

Stability region by a conformal map:

Let $G = 1 + z + \frac{z^2}{2}$ $\qquad z = \lambda h$

$\therefore z = -1 \pm \sqrt{2G - 1}$



Stability Region of the Trapezoidal Scheme: $Y_{k+1} = Y_k + \frac{h}{2}\left[f(x_n, Y_n) + f(x_{n+1}, Y_{n+1})\right]$

Consider the model problem $y' = \lambda y \quad y(0) = y_0$.

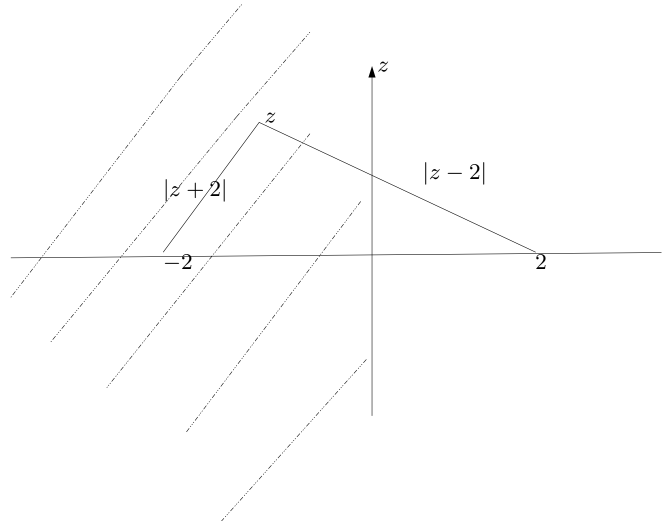
$$\begin{aligned} Y_{k+1} &= Y_k + \frac{h}{2}\left[\lambda Y_k + \lambda Y_{k+1}\right] \\ \therefore & \left(1 - \frac{h\lambda}{2}\right) Y_{k+1} = \left(1 + \frac{h\lambda}{2}\right) Y_k \\ Y_{k+1} &= \frac{(1 + h\lambda/2)}{(1 - h\lambda/2)} Y_k = G(h\lambda) Y_k \text{ where } G(h\lambda) = \frac{1 + h\lambda/2}{1 - h\lambda/2} \end{aligned}$$

- **Note**: $e^z \simeq G(z) = \dfrac{1 + z/2}{1 - z/2}$ is the $(1,1)$ Padé Approximation of e^z.

$$e^z = \frac{1 + a_1 z + \ldots + a_n z^m}{1 + b_1 z + \ldots + b_n z^n} \text{ is the } (m,n) \text{ Padé approximant.}$$

Stability: For stability we require that $|G(h\lambda)| < 1$.

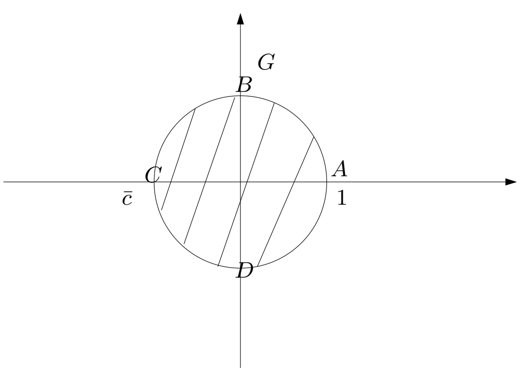
$$\begin{aligned} G(z) &= \frac{1 + z/2}{1 - z/2} = \frac{2+z}{2-z} \\ 1 > |G(z)| &= \frac{|z+2|}{|z-2|} \Rightarrow |z-2| > |z+2| \end{aligned}$$

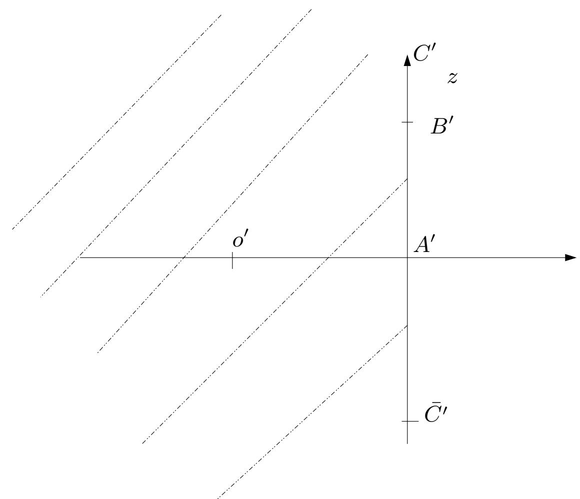


• The Trapezium Rule is A-stable but it is more expensive to compute with it since it is implicit.

Alternatively using a conformal map:

$$G = \frac{2+z}{2-z} \Rightarrow (2-z)G = 2 + z \Rightarrow 2(G-1) = z(1+G) \Rightarrow z = 2\frac{(G-1)}{(G+1)}.$$



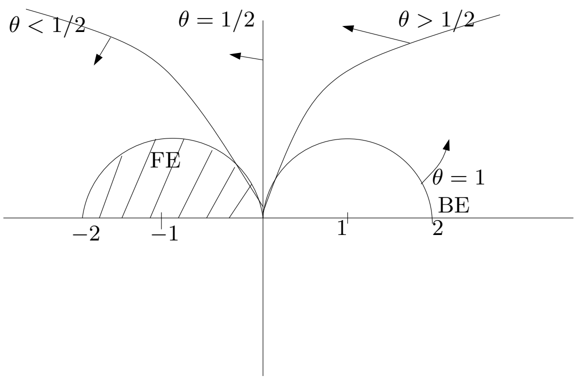


$$\begin{aligned} O:& \quad G = 0 \Rightarrow z = -2 \\ A:& \quad G = 1 \Rightarrow z = 0 \\ B:& \quad G = i \Rightarrow z = 2\left(\frac{i-1}{i+1}\right) \\ C:& \quad G \to -1_+ \Rightarrow z \to i\infty \\ \overline{C}:& \quad G \to -1_- \Rightarrow z \to -i\infty \\ D:& \quad G = -i \Rightarrow z = -2i \end{aligned}$$

General Implicit Method – θ method

$$Y_{n+1} = Y_n + h\left[(1-\theta) f(x_n, Y_n) + \theta f(x_{n+1}, Y_{n+1})\right]$$

$\theta = 0:$	Forward Euler / Explicit Euler	$G = 1+z$	$(1,0)$ Padé approx. to $e^{z/2}$
$\theta = 1/2:$	Trapezoidal Rule	$G = \frac{1+z/2}{1-z/2}$	$(1,1)$ Padé approx. to $e^{z/2}$
$\theta = 1:$	Backward Euler /Implicit Euler	$G = \frac{1}{1-z}$	$(0,1)$ Padé approx. to $e^{z/2}$.



Truncation Error: Let $y'_n = f(x_n, y_n)$

$$\begin{aligned} T_n(h) &= \frac{y_{n+1} - y_n}{h} - \left[(1-\theta) f(x_n, y_n) + \theta f(x_{n+1}, y_{n+1})\right] \\ &= \frac{y_n + h y'_n + \frac{h^2}{2} y''_n + \frac{h^3}{3!} y'''_n + \ldots - y_n}{h} - \left[(1-\theta) f_n + \theta\left[f_n + h(f_x + f f_y)\Big|_{x_n} + O(h^2)\right]\right] \\ &= (y'_n - f_n) + \frac{h}{2}(y''_n - 2\theta(f_x + f f_y)\Big|_{x_n}) + O(h^2) \\ &= \begin{cases} O(h) & \theta \neq 1/2 \\ O(h^2) & \theta = 1/2. \end{cases} \end{aligned}$$

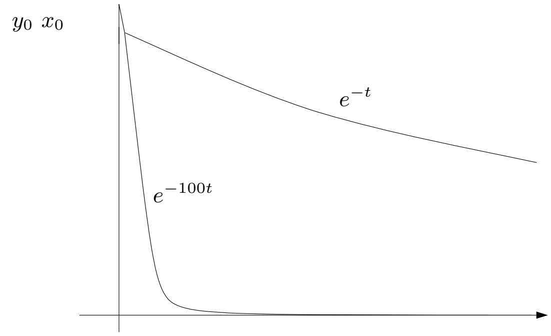
Note:

- An explicit method cannot be A-stable.
- The order of an A-stable implicit method cannot exceed 2.
- The second order A-stable implicit method with the smallest error constant is the trapezoidal rule.
- Looks like the TR is a winner but there is an important class of problems for which TR gives poor results – stiff systems.

Stiff Systems:

Example:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -100 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{array}{l} x = e^{-t}x_0 \\ y = e^{-100t}y_0 \end{array}$$



• **FE**: If we were to use the FE method in the useful regime we would requre $-2 < h\lambda_k < 0$

$$\lambda_1 = -1 \quad \Rightarrow \quad h < 2$$
$$\lambda_2 = -100 \quad \Rightarrow \quad h < 1/50$$

We do not particularly care about y since it decays to zero very rapidly but we are more interested in x which persists much longer. But to compute the system stably we woud need very small time-steps – bad news; it will take forever.

• What about using the **Trapezoidal Rule** so we don't have to worry about the timestep?

Say $Re(\lambda) \to -\infty$ and let us look at $G(z) = \frac{1+z/2}{1-z/2}$ in the case $Re(\lambda) \to -\infty$.

Let $z = \alpha + i\beta$ and let β be fixed.

$$|G(z)| = \frac{|1+z/2|}{|1-z/2|} = \frac{\sqrt{(1+\alpha/2)^2 + (\beta/2)^2}}{\sqrt{(1-\alpha/2)^2 + (\beta/2)^2}} \quad \xrightarrow{\alpha \to -\infty} \quad 1$$

solution will oscillate but will not decay!

But e^z, which $G(z)$ is supposed to approximate, is such that $e^z \to 0$ as $Re(z) \to -\infty$.

L-Stability: A numerical method for which $G(z) \to 0$ as $Re(z) \to -\infty$ is said to be L-stable or has strong decay.

•**Example of an L-stable method**: – the Backward Euler Scheme: (BE)

$$Y_{n+1} = Y_n + hf(x_{n+1}, Y_{n+1})$$

For model problem:

$$\begin{aligned} Y_{n+1} &= Y_n + h\lambda Y_{n+1} \\ Y_{n+1} &= \frac{1}{(1-h\lambda)} Y_n = G(h\lambda)Y_n \end{aligned}$$

$$G(z) = \frac{1}{1-z} \qquad z = 1 - \frac{1}{G}$$

$$G(z) \to 0 \quad \text{as} \quad Re(z) \to -\infty$$

so BE is L-stable.