Eg: For the Euler Method $\dfrac{Y_{n+1}-Y_n}{h} = f(x_n, Y_n)$ the truncation error is:

$$T_n = \frac{y_{n+1}-y_n}{h} - f(x_n, y_n) = \frac{y_n + hy_n' + \frac{h^2}{2}y_n'' + \ldots - y_n}{h} - f(x_n, y_n)) = O(h)$$

Consistency:

A difference equation is **consistent** with a differential equation if the truncation error $\to 0$ as $h \to 0$.

Note: Consistency $\Rightarrow$ Difference equation $\overset{h\to 0}{\longrightarrow}$ Differential equation you want to solve and not some other differential equation

Convergence:

A difference method converges with order p if for h sufficiently small

$$|Y_n(h) - y(nh)| \le C(b-a)h^p \quad \text{for} \quad nh < (b-a).$$

Theorem: *The FE method converges with first order accuracy on* $[a, b]$

Proof: Let $e_n = Y_n - y_n$ and let $L = \max\left|\frac{\partial f}{\partial y}\right| \qquad f \in C'[a, b]$

$$\begin{aligned}
Y_{n+1} &= Y_n + hf(x_n, Y_n) \\
y_{n+1} &= y_n + hf(x_n, y_n) + \frac{h^2}{2}y''(\xi) \\
\text{Subtract} \quad e_{n+1} &= e_n + h\left\{f(x_n, Y_n) - f(x_n, y_n)\right\} - \frac{h^2}{2}y''(\xi) \\
\text{Since} \quad f &\in C^1 \text{ using the MVT} \\
& |f(x_n, Y_n) - f(x_n, y_n)| = |f_y(x_n, \tilde{y})|\,|Y_n - y_n| \le L|e_n| \\
\text{Also let} \quad y'' &= f_x + f\,f_y \in C^0[a, b] \Rightarrow \text{ there exists an } M : \\
& \qquad |y''(\xi)|/2 \le M \;\forall\, \xi \in [a, b] \\
\therefore \quad & \boxed{|e_{n+1}| \le (1+hL)|e_n| + Mh^2, \quad e_0 = 0} \quad \text{A Difference inequality for the error.}
\end{aligned}$$

Mh^2 = Local truncation error.

Consider the difference equation:

$$E_{n+1} = (1+hL)E_n + Mh^2 \qquad E_0 = 0 \qquad (*)$$

Claim: $|e_n| \le E_n$

PF: Induction $n = 0$ trivial since $E_0 = 0 = e_0$

$$|e_{n+1}| = (1+hL)|e_n| + Mh^2 \le (1+hL)E_n + Mh^2 = E_{n+1}$$

■

Solution to (*)

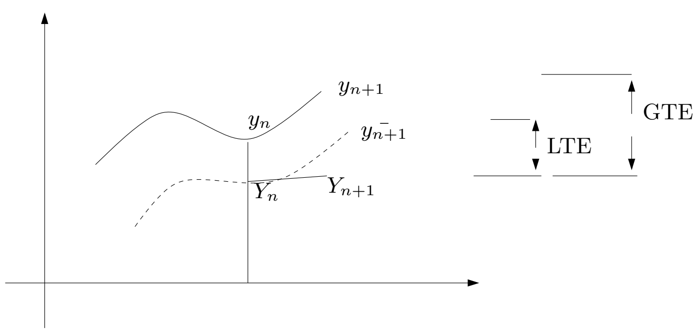
Homog: $E_n = A(1+hL)^n$
Particular: $E_n = D \Rightarrow D = (1+hL)D + Mh^2 \rightarrow D = \frac{-Mh^2}{hL} = \frac{-M}{L}h$

$$
\begin{aligned}
\therefore E_n &= A(1+hL)^n - Mh/L \\
E_0 &= 0 = A - Mh/L \\
\therefore E_n &= \frac{Mh}{L}\left\{(1+hL)^n - 1\right\} \\
\therefore |e_n| &\leq \frac{Mh}{L}\left\{(1+hL)^n - 1\right\} \qquad e^x \geq 1 + x \\
&\leq \frac{Mh}{L}\left(e^{hLn} - 1\right) \qquad \text{since } g(x) = e^x - 1 - x \text{ is increasing on [a,b].}
\end{aligned}
$$

$$\boxed{|e_n| \leq \frac{M}{L}\left(e^{L(b-a)} - 1\right) h = O(h)} \longleftarrow \text{Global truncation error}$$

Notes:

1. Local truncation error (LTE): $O(h^2)$ & Global truncation error (GTE) $O(h)$



$\boxed{\text{Essentially } NO(h^2) = O(h)}$

2. Since we are primarily interested in the GTE, many authors only define the truncation error as was done above and refer to it as the truncation error.

How about a general convergence theory?

– We don't want to have to repeat the above theorem for each new scheme.

– The key elements of a convergence theory to prove convergence of the solution to a difference scheme to the solution to the initial value problem $y' = f(x, y) \quad y(0) = y$.

- Consistency - truncation error is small enough that as $h \to 0$, the difference equation $\to$ the differential equation you want to solve and not some other equation.
- Stability – Roundoff errors do not grow as the solution evolves.

Eg: Consider the IVP $y' = \lambda y \quad y(0) = 1$ with the solution $y = e^{\lambda x}$ and the difference scheme

$$Y_{n+2} - 2Y_{n+1} + Y_n = \frac{h}{2}\left[f(x_{n+2}, Y_{n+2}) - f(x_n, Y_n)\right].$$

For this problem

$$\boxed{Y_{n+2} - 2Y_{n+1} + Y_n = \tfrac{h\lambda}{2}\left[Y_{n+2} - Y_n\right], \text{ with } IC\; Y_0 = 1, Y_1 = 1.}$$

Plug in solution to $y' = \lambda y$

$$\begin{aligned}
T_n &= \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} - \frac{\lambda}{2}\frac{[y_{n+2} - y_n]}{h} \\
&= \frac{y_n + 2hy_n' + \frac{(2h)^2}{2}y_n'' + 0(h^3) - 2\left\{y_n + hy_n' + \frac{h^2}{2}y_n'' + 0(h^3)\right\} + y_n}{h^2} - \frac{\lambda}{2}\,\frac{\left\{y_n + (2h)y_n' + \frac{(2h)^2}{2}y_n'' + \ldots - y_n\right\}}{h} \\
&= (y_n'' - \lambda y_n') + O(h) \qquad \text{so the method is consistent.}
\end{aligned}$$

Let us see how small perturbations to the solution will propagate.

1) We look for a solution to the homogeneous difference equation (*)

$$\begin{aligned}
Y_n &= \theta^n \Rightarrow \left(1 - \frac{h\lambda}{2}\right)\theta^2 - 2\theta + \left(1 + \frac{h\lambda}{2}\right) = 0 \\
\text{with roots } \theta &= \frac{1 \pm \frac{1}{2}\sqrt{4 - 4\left(1 - (\frac{h\lambda}{2})^2\right)}}{(1 - h\lambda/2)} = \frac{1 \pm (h\lambda)/2}{1 - (h\lambda/2)} = 1 \text{ or } \frac{1 + (h\lambda)/2}{1 - (h\lambda)/2}
\end{aligned}$$

2) Let us assume that the perturbations take on the form $Y_0 = \delta$, $Y_1 = \delta$ and that

$$Y_{n+2} - 2Y_{n+1} + Y_n = \frac{h\lambda}{2}\left[Y_{n+2} - Y_n\right] + h\delta.$$

Let us see how these perturbations will propagate:

Particular solution: Assume $Y_n = C$ a constant

$$\therefore \qquad C - 2C + C = \frac{h\lambda}{2}[C - C] + h\delta$$

We note that C is a solution to the homogeneous equation so we look for a particular solution of the form

$$\begin{aligned}
Y_n &= nc \\
(n+2)c &- 2(n+1)c + nc = \frac{h\lambda}{2}[(n+2)c - nc] + h\delta \\
&\therefore\; h\lambda c + h\delta = 0 \\
&\quad c = -\delta/\lambda \\
\therefore Y_n &= -n\delta/\lambda \qquad \text{is a particular solution.}
\end{aligned}$$

General solution

$$\begin{aligned}
Y_n &= A + B\left(\frac{1 + h\lambda/2}{1 - h\lambda/2}\right)^n - n\delta/\lambda \\
Y_0 &= A + B = \delta \\
Y_1 &= A + B\left(\frac{1 + h\lambda/2}{1 - h\lambda/2}\right) - \delta/\lambda = \delta
\end{aligned}$$

$$B\left\{1-\left(\frac{1+h\lambda/2}{1-h\lambda/2}\right)\right\} = -\delta/\lambda$$
$$B(h\lambda) = -\delta/\lambda$$
$$\therefore B = -\delta/h\lambda^2 \qquad\qquad A = \delta - B = \delta(1+1/h\lambda^2)$$
$$\therefore Y_n \;=\; \delta\left(1+\frac{1}{h\lambda^2}\right) - \frac{\delta}{h\lambda^2}\left(\frac{1+h\lambda/2}{1-h\lambda/2}\right)^n - \frac{n\delta}{\lambda}.$$

Now as $h \to 0$ and $n \to \infty$ in such a way that $nh = x$ constant we observe that $Y_n \to \infty$

Eg: **Stability**: Consider the model problem

$$y' = \lambda y \qquad y(0) = 1 \qquad \text{with } y = e^{\lambda x}$$

and consider the solution generated by the Euler scheme

$$Y_{n+1} - Y_n = h\lambda Y_n$$

and consider perturbations of the form $Y_0 = 1 + \delta$ and look at

$$Y_{n+1} - Y_n = h\lambda Y_n + h\delta \qquad Y_0 = \delta \quad \text{(Subtract out the exact solution to the DCE)}$$

Homogeneous eq: $Y_n = \theta^n$

$$\theta = (1 + h\lambda)$$

Particular solution: $Y_n = c \to c - c = h\lambda c + h\delta \to c = -\delta/\lambda$

$\therefore$ General solution is

$$Y_n = A(1+h\lambda)^n - \delta/\lambda \qquad Y_0 = \delta \to A - \delta/\lambda = \delta \to A = \frac{\delta}{\lambda}(1+\lambda)$$

If $\lambda < 0$ and $h \to 0$, $n \to \infty$ in such a way that $hn = x$ a constant then

$$\begin{aligned} Y_n &= A\left(1 - \frac{x|\lambda|}{n}\right)^n - \delta/\lambda \\ &= A\exp\left\{-x|\lambda|\frac{\ln\left(1-\frac{x|\lambda|}{n}\right)}{\frac{-x|\lambda|}{n}}\right\} - \delta/\lambda \\ &\stackrel{n\to\infty}{=} \frac{\delta}{\lambda}(1+\lambda)e^{-x|\lambda|} - \frac{\delta}{\lambda} \qquad \text{which is bounded.} \end{aligned}$$