Initial Value ODE

Consider the system of nonlinear ODE with prescribed initial value.

$$\left.\begin{array}{ll} y' = f(x, y(x)) & \\ y(a) = y_0 & \text{initial conditions} \end{array}\right\} (1) \qquad y \in IR^n$$

Note:

1. If $f \in C^1$ then (1) has a unique solution
2. The behavior of errors in the numerical solution of (1) is related to the behavior of the linearized eq: Let $\bar{y}(x)$ be some nominal solution and δy a perturbation . Then

$$\begin{aligned} y(x) &= \bar{y}(x) + \delta y(x) \\ y' = (\bar{y} + \delta y)' &= f(x, \bar{y} + \delta y) \\ \bar{y}' + \delta y' &= f(x, \bar{y}) + \frac{\partial f}{\partial y}(x, \bar{y})\delta y \\ \delta y' &= \frac{\partial f}{\partial y}(x, \bar{y})\delta y = A(x)\delta y \qquad (*) \\ \frac{\partial f}{\partial y}(x, \bar{y}) &= \text{the Jacobian matrix of } f(x, y). \end{aligned}$$

3. The Model Problem:

   Assume $A(x) = A$ (a constant in time) and that A has N distinct eigenvalues λ_j and N independent eigenvectors. Then by making a change of variables

$$\delta y = Pz \qquad P = [v_1|v_2|\dots|v_N]$$

   We can rewrite (*) in the form

$$z' = Dz$$

   where

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}$$

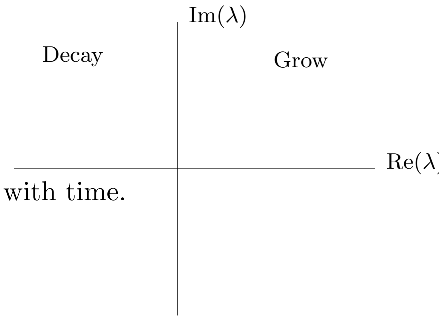
   So the equations for z are decoupled into the form

$$z_h' = \lambda_j z \qquad j = 1, \dots, N$$

Scalar Model Problem:
We consider the scalar model problem
$y' = \lambda y, \qquad y(0) = y_0$
with the exact solution $y = y_0 e^{\lambda x}$.

Note: If $Re(\lambda) > 0$ solutions grow exponentially.
If $Re(\lambda) < 0$ solutions decay exponentially.

Consequence for system of ODE:



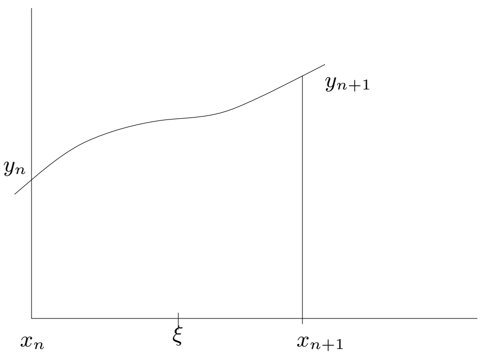
- If $\dfrac{\partial f}{\partial y}(x, y(x)) = A(x)$ has eigenvalues all of whose real parts are negative then errors will decay exponentially with time.
- If any one eigenvalue of $A(x)$ has a positive real part, then errors will grow exponentially with time.

Schemes to solve the scalar initial value problem:

Consider

$$\begin{aligned} y' &= f(x,y) \qquad\qquad y \in \mathbb{R} \\ y(0) &= y_0 \end{aligned}$$

1. **The Taylor Series Method**:



$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \cdots + \frac{h^r}{r!}y^{(r)}(x_n) + \frac{h^{r+1}}{(r+1)!}\, y^{(r+1)}(\xi)$$

Now $\quad y' = f(x, y(x))$

$\quad y'' = f_x(x,y) + f_y y' = f_x + f f_y$

Eg: $\quad y' = \lambda y, \quad y(0) = y_0$

$\quad y'' = \lambda y' = \lambda^2 y$

$\quad y^{(r)} = \lambda y^{(r-1)} = \cdots = \lambda^r y$

$$\therefore \quad y_{n+1} = y_n + h\lambda y_n + \cdots + \frac{(h\lambda)^r}{r!} y_n + \cdots$$

$$= \left(1 + (h\lambda) + \cdots + \frac{(h\lambda)^r}{r!} + \cdots\right) y_n = e^{\lambda h} y_n$$

Note:

(1) By truncating the Taylor series at the rth term, we obtain an approximation of $O(h^r)$– derivative evaluation tedious.

(2) The accuracy of a numerical scheme is determined by the number of terms of agreement with the Taylor Series when the exact solution of the ODE is substituted into the difference equation.

(3) Many numerical schemes can be interpreted as giving different approximations to $e^{\lambda h}$ when they are applied to the model problem.

2. **The forward Euler Method – A prototype ODE solver**:

Idea: Truncate the Taylor series after the linear term and avoid having to take higher derivatives.

$$\begin{aligned} y_{n+1} &= y_n + h y_n' + O(h^2) \\ &= y_n + h f(x_n, y_n) + O(h^2) \end{aligned}$$

Euler's Method:

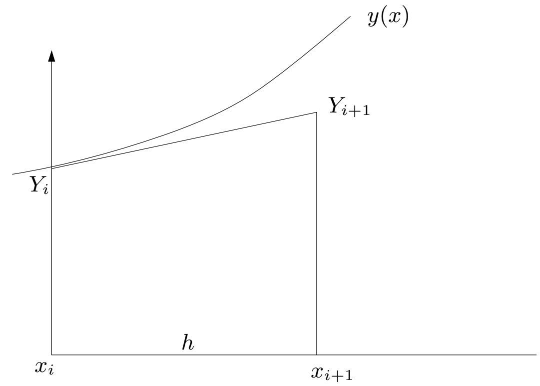
$$\begin{aligned} Y_{n+1} &= Y_n + h f(x_n, Y_n) \\ Y_0 &= y_0 \end{aligned}$$

Where $Y_n \simeq y(x_n)$
Difference equation

Alternative Derivation 1: Using the forward difference approx. to y':

$$\begin{aligned} \frac{y_{i+1} - y_i}{h} &= y_i' + Mh \\ \frac{Y_{i+1} - Y_i}{h} &= f(x_i, Y_i) \end{aligned}$$

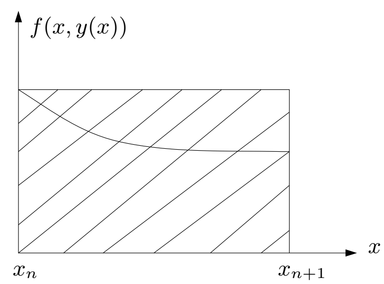
M depends on y''.



Alternative Derivation 2:

$$\begin{aligned} y' &= f(x, y(x)) \\ y(x_{n+1}) &= y(x_n) + \int_{x_n}^{x_{n+1}} f(s, y(s))\, ds \\ Y_{n+1} &= Y_n + h\, f(x_n, Y_n) \end{aligned}$$

Left hand approximate integration



NOTE:

1. The Forward Euler (FE) is **explicit** because all the information to proceed from the n^{th} step to the $(n+1)^{\text{th}}$ step is known. Contrast this with $Y_{n+1} = Y_n + h\, f(x_{n+1}, Y_{n+1})$ which involves solving a nonlinear equation at each time step.

2. The Euler method involves a difference equation that can be thought of as a model for $y' = f(x, y)$.

Truncation error:

The **Truncation Error** is the term that remains when you plug the exact solution to $y' = f(x, y)$ into the difference scheme.