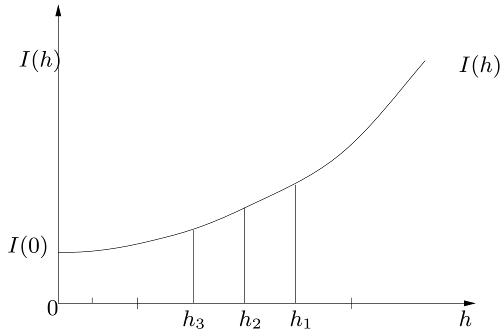
Richardson Extrapolation:

Exploiting the error estimate to get an improved approximation:



Because we have an error estimate for the Trapezium rule of the form:

$$\begin{aligned} I(h) &= I(0) + c_2 h^2 + c_4 h^4 \qquad I(4h) = I(0) + c_2 16h^2 + c_4 256 h^4 \\ I(2h) &= I(0) + c_2 4h^2 + c_4 16 h^4 \end{aligned}$$

eliminate the c and get an improved estimate:

$$\begin{aligned} 4I(h) - I(2h) &= 3I(0) - 12c_4 h^4 \\ \therefore \quad \frac{4I(h) - I(2h)}{3} &= I(0) - 4c_4 h^4 \end{aligned}$$

Eg:

$$\begin{aligned} I &= \int_0^1 \sin \pi x \, dx \\ I(1/4) &= 0.60355339059327 \\ I(1/8) &= 0.62841743651573 \\ \therefore \quad \frac{4I(1/8) - I(1/4)}{3} &= 0.6370545 \approx 0.63661977 \end{aligned}$$

We can continue with this process using the recursion

$$\begin{aligned} a_s^{(1)} &= I(h_s) \qquad s = 1, \ldots, k \\ a_s^{(m)} &= a_{s+1}^{(m-1)} + \frac{\left(a_{s+1}^{(m-1)} - a_s^{(m-1)}\right)}{(h_s/h_{s+m-1})^\gamma - 1} \qquad \begin{array}{c} s = 1, \ldots, k-m+1 \\ m = 2, \ldots, k \end{array} \end{aligned}$$

and where expansion for the error is of the form

$$I(h) = I(0) + \sum_{k=1}^{N} C_{\gamma k} (h^\gamma)^k$$

Note: Richardson extrapolation combined with adaptive integration is known as Romberg integration.

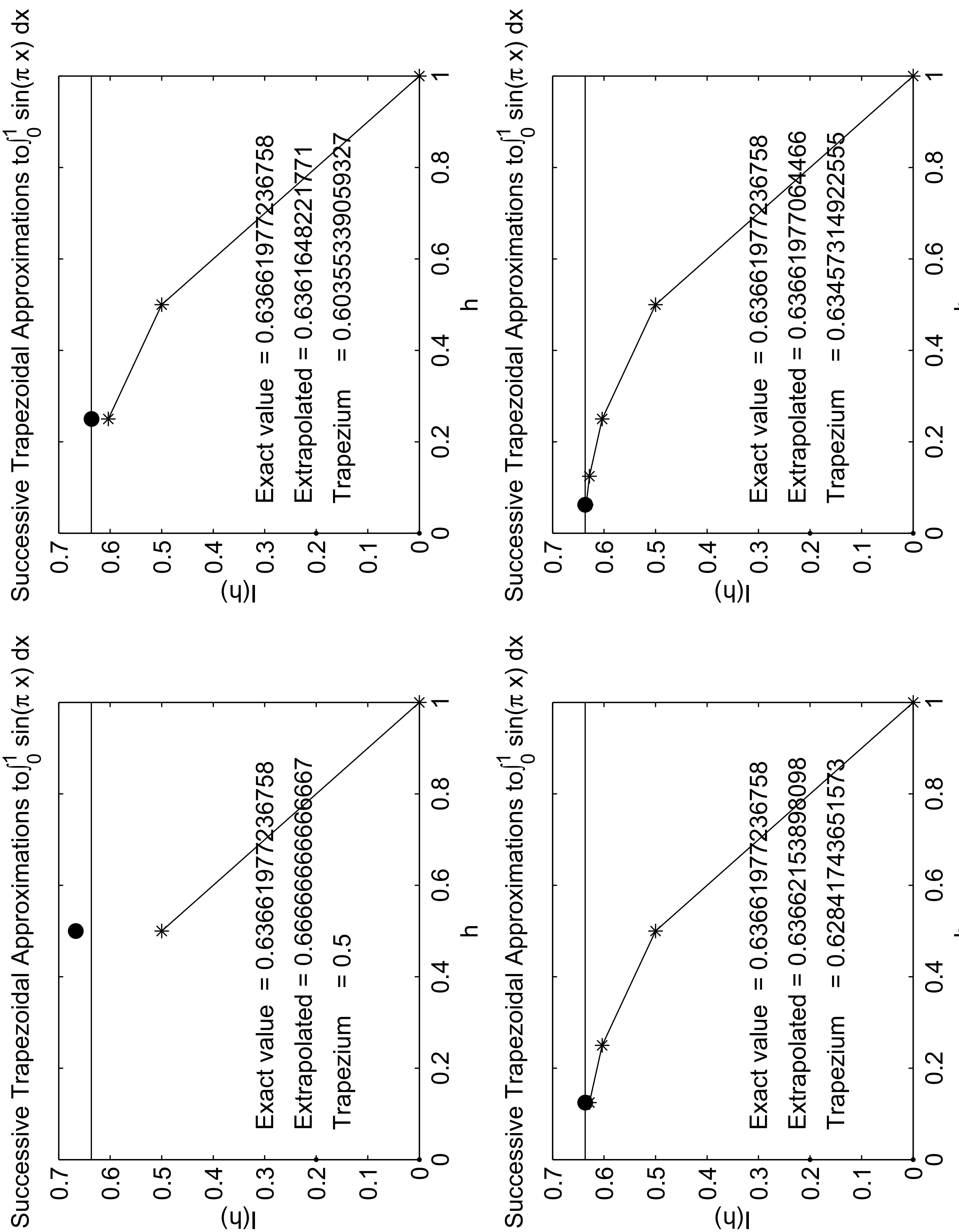
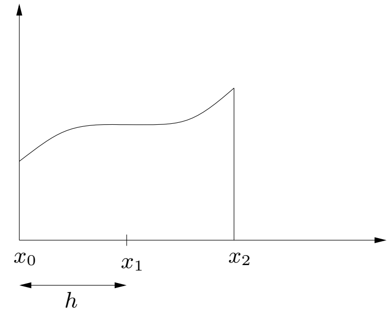


Figure 1: Plot of successive Trapezoidal approximations as well as the extrapolated values

Simpson's Rule:



$$p = \frac{x - x_0}{h}, \quad x = x_0 + ph, \quad dx = h\,dp$$

Recall

$$\begin{aligned}
E^p f_0 &= (1+\Delta)^p f_0 \\
&= \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \dots\right) f_0 \\
E^p f_0 &\approx \left(1 + p\Delta + \frac{p(p-1)}{2}\Delta^2\right) f_0 \qquad \text{for a polynomial of degree 2.} \\
\int_{x_0}^{x_2} f(x)\,dx &= h\int_0^2 \left(f_0 + p\Delta f_0 + \frac{1}{2}(p^2 - p)\Delta^2 f_0\right) dp \\
&= h\left\{p f_0 + \frac{p^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 f_0\right\}_0^2 \\
&= h\left\{2f_0 + 2(f_1 - f_0) + \frac{1}{2}\left(\frac{8}{3} - 2\right)(f_2 - 2f_1 + f_0)\right\} \\
&= h\left\{2f_1 + \frac{1}{3}f_2 - \frac{2f_1}{3} + \frac{1}{3}f_0\right\}
\end{aligned}$$

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left\{f_0 + 4f_1 + f_2\right\}} \qquad \text{Simpson's Rule (Requires 2 intervals).}$$

Error involved:

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,dx &= \int_{x_0}^{x_2} P_2(x)\,dx + \frac{f^{(3)}(\xi)}{3!}\int_{x_0}^{x_2}(x-x_0)(x-x_1)(x-x_2)\,dx \\
\chi &= x - x_1 \quad x = x_1 + \chi \\
\int_{x_0}^{x_2}(x-x_0)(x-x_1)(x-x_2)\,dx &= \int_{-h}^{h}(\chi+h)\chi(\chi-h)d\chi = \int_{-h}^{h}\chi^3 - h^2\chi\,d\chi = 0. \\
\therefore \int_{x_0}^{x_2} f(x)\,dx &= \frac{h}{3}\left\{f_0 + 4f_1 + f_2\right\} + \int_{x_0}^{x_2} f[x_0,x_1,x_2,x](x-x_0)(x-x_1)(x-x_2)\,dx
\end{aligned}$$

Now

$$f[x_0,x_1,x_2,x] - f[x_0,x_1,x_2,x_3] = (x-x_3)f[x_0,x_1,x_2,x_3,x]$$

$$\therefore \int_{x_0}^{x_2} f[x_0,x_1,x_2,x](x-x_0)(x-x_1)(x-x_3)dx = \int_{x_0}^{x_2} f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)\,dx$$

$$+\int_{x_0}^{x_2} f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\;dx$$

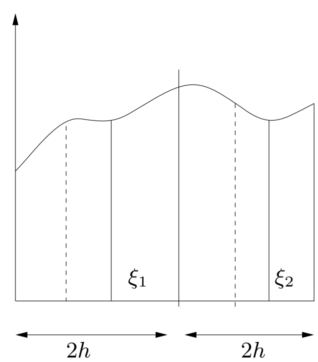
$$\int_{x_0}^{x_2} f(x)dx = S + \frac{f^{(4)}(\xi)}{4!}\int_{x_0}^{x_2}(x-x_0)(x-x_1)(x-x_2)(x-x_3)\;dx \qquad \text{choose } x_3 = x_1$$

$$\int_{x_0}^{x_2}(x-x_0)(x-x_1)^2(x-x_2)\;dx = \int_{-h}^{h}\left(\chi^2-h^2\right)\chi^2 d\chi = \frac{2h^5}{5}-\frac{2h^5}{3} = h^5\frac{6-10}{15} = -\frac{4h^5}{15}$$

$$\boxed{\therefore \int_{x_0}^{x_2} f(x)\;dx = \frac{h}{3}\left\{f_0+4f_1+f_2\right\} - \frac{f^{(4)}(\xi)}{90}h^5}$$

Composite Rule:

$$\int_a^b f(x)dx \quad = \quad \frac{h}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+4f(x_{N-1})+f(x_N)\right] - \frac{h^5}{90}\sum_{k=1}^{N/2} f^{(4)}(\xi_k)$$

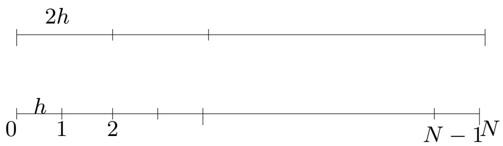


$$\begin{aligned}
&= S - \frac{h^4}{180}(2h)\sum_{k=1}^{N/2} f^{(4)}(\xi_k) \\
&= S - \frac{h^4}{180}\left(f^{(3)}(b)-f^{(3)}(a)\right) \\
&= S - \frac{h^4}{180}(b-a)f^{(4)}(\xi)
\end{aligned}$$

$$\begin{aligned}
2h\sum_{k=1}^{N/2} f^{(4)}(\xi_k) &\approx \int_a^b f^{(4)}(x)\;dx \\
&= f^{(3)}(b)-f^{(3)}(a)
\end{aligned}$$

Neat interpretation of the first extrapolation formula for the trapezium rule:

$$I(0) = \frac{4}{3}I(h) - \frac{1}{3}I(2h) + O(h^4)$$



$$\begin{aligned} I(0) &= \frac{4}{3}\cdot\frac{h}{2}\left\{f_0 + 2f_1 + \cdots + 2f_{N-1} + f_N\right\} - \frac{1}{3}\frac{\cancel{2}h}{\cancel{2}}\left\{f_0 + 2f_2 + \cdots + 2f_{N-2} + f_N\right\} \\ &= \frac{h}{3}\left\{f_0 + 4f_1 + 2f_2 + \cdots + 2f_{N-2} + 4f_{N-1} + f_N\right\} \qquad \text{just Simpson's Rule.} \end{aligned}$$

Closed Newton-Cotes Formulae:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12}\,f^{(2)}(\xi) \qquad \text{Trapezium rule} \qquad \xi \in (x_0, x_1)$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{h^5}{90}\,f^{(4)}(\xi) \qquad \text{Simpson's rule} \qquad \xi \in (x_0, x_2)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80}\,f^{(4)}(\xi) \qquad \xi \in (x_0, x_3)$$

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) - \frac{8h^7}{945}\,f^{(6)}(\xi) \qquad \xi \in (x_1, x_4)$$