0.3 Finite elements in 2D

Interpolation in 2D:

$$u^h(x,y) = \sum_{i=1}^{N} N_i(x,y) u_i \qquad\qquad N_i(x_j, y_i) = \delta_{ij}$$

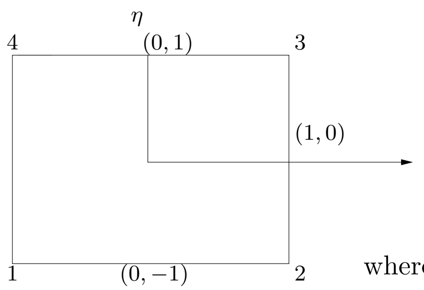
How do we construct basis functions? Can we still map onto canonical elements?

ISOPARAMETRIC ELEMENTS (most commonly used)

IDEA 1: Same basis functions are used to transform from canonical elements to actual elements in the mesh as those that are used to represent the unknown solution.

IDEA 2: Use products of 1D basis functions to construct 2D basis functions.

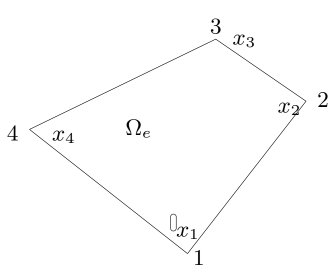
1. Bilinear elements:

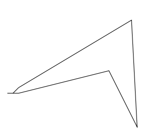


$$\begin{aligned} \mathbf{x}(\xi) &= \sum_{a=1}^{4} \mathbf{x}_a N_a(\boldsymbol{\xi}) \qquad \text{where } N_a(\boldsymbol{\xi}_b) = \delta_{ab} \qquad \begin{matrix} \boldsymbol{\xi} = (\xi, \eta) \\ \mathbf{x}_a = (x_a, y_a) \end{matrix} \\ u^h(\boldsymbol{\xi}) &= \sum_{a=1}^{4} u_a N_a(\boldsymbol{\xi}) \end{aligned}$$

where

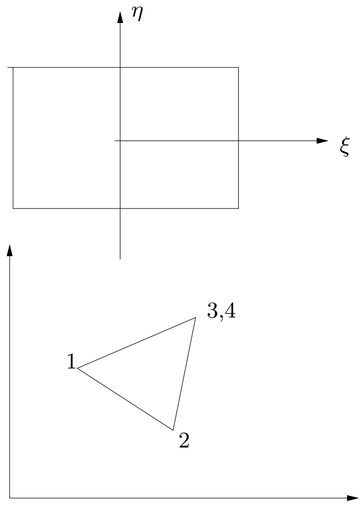
$$\begin{aligned} N_a(\boldsymbol{\xi}) &= \frac{1}{4}(1+\xi^a\xi)(1+\eta^a\eta) = N_a(\xi)N_a(\eta) \\ N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta) = N_1(\xi)N_1(\eta) \\ N_2(\xi,\eta) &= N_2(\xi)N_1(\eta) = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) &= N_2(\xi)N_2(\eta) = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) &= N_1(\xi)N_2(\eta) = \frac{1}{4}(1-\xi)(1+\eta) \end{aligned}$$





AVOID BAD DISTORTIONS

2. Triangle as a degenerate rectangle:



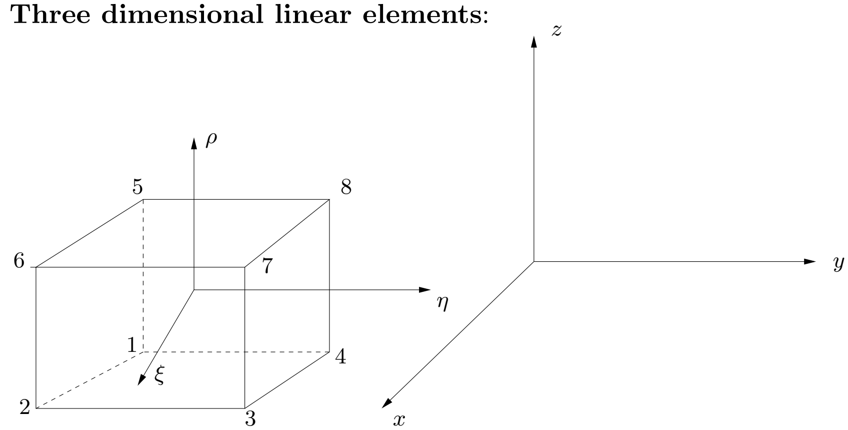
$$\mathbf{x} = \sum_{a=1}^{4} N_a \mathbf{x}^a = N_1\mathbf{x}_1 + N_2\mathbf{x}_2 + \{N_3 + N_4\}\mathbf{x}_3$$
$$= \sum_{a=1}^{3} \overline{N}_{\dot{a}}(\boldsymbol{\xi}) x_a$$

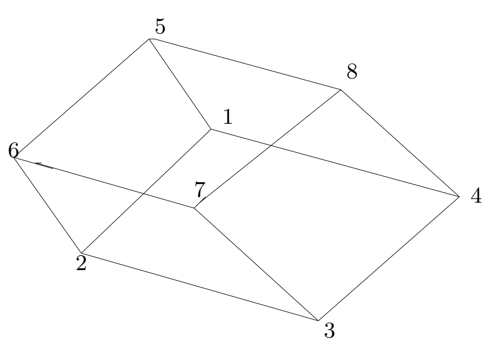
where

$$\overline{N}_a = N_a \qquad a = 1, 2, \qquad \overline{N}_3(\boldsymbol{\xi}) = N_3(\xi,\eta) + N_4(\xi,\eta)$$
$$= \frac{1}{4}(1+\xi)(1+\eta) + \frac{1}{4}(1-\xi)(1+\eta)$$
$$= \frac{1}{2}(1+\eta)$$

(-) Not very good because derivatives can be piecewise constant.
(+) Triangular tessellations are very easy.

3. Three dimensional linear elements:

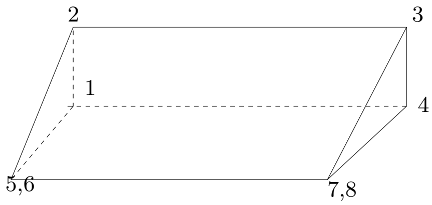




$$\mathbf{x} = \sum_{d=1}^{8} N_a(\boldsymbol{\xi})\mathbf{x}_a$$

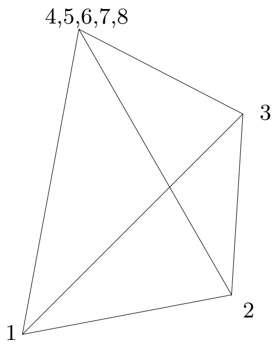
$$
\begin{aligned}
u^h &= \sum_{a=1}^{8} N_a(\xi) u_a \\
N_a(\xi) &= N_a(\xi) N_a(\eta) N_a(\rho) = \frac{1}{8}(1+\xi_a \xi)(1+\eta_a \eta)(1+\rho_a \rho)
\end{aligned}
$$

Wedge elements:



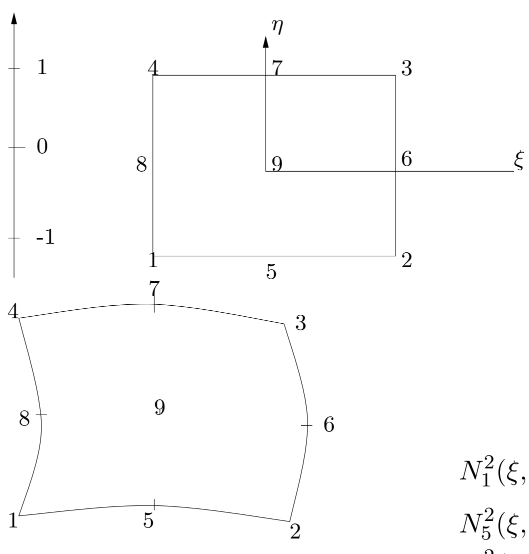
$$
\begin{aligned}
\overline{N}_5 &= N_5 + N_6 \\
\overline{N}_6 &= N_7 + N_8
\end{aligned}
$$

Tetrahedral elements:



$$\overline{N}_4 = N_4 + \ldots + N_8$$

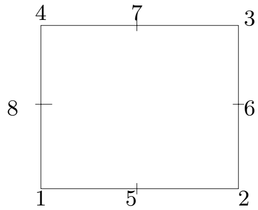
4. Biquadratic elements:



9-node Lagrange ELT.

$$
\begin{aligned}
N_1^2(\xi,\eta) &= N_1^2(\xi) N_1^2(\eta) = \frac{1}{4}\xi(\xi-1)\eta(\eta-1) \\
N_5^2(\xi,\eta) &= N_2^2(\xi) N_1^2(\eta) \\
N_9^2(\xi,\eta) &= N_2^2(\xi) N_2^2(\eta)
\end{aligned}
$$

8-Node serendipity element:



$$
\begin{aligned}
N_a^s(\xi,\eta) &= \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)(\xi_a\xi+\eta_a\eta-1) \qquad && a=1,2,3,4 \\
N_a^s(\xi,\eta) &= \frac{\xi_a^2}{2}(1+\xi_a\xi)(1-\eta^2)+\frac{\eta_a^2}{2}(1+\eta_a\eta)(1-\xi^2) \qquad && a=5,6,7,8
\end{aligned}
$$