COARSE QUANTIZATION FOR RANDOM INTERLEAVED SAMPLING OF BANDLIMITED SIGNALS

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Abstract. The compatibility of unsynchronized interleaved uniform sampling with Sigma-Delta analog-to-digital conversion is investigated. Let f be a bandlimited signal that is sampled on a collection of N interleaved grids $\{kT+T_n\}_{k\in\mathbb{Z}}$ with offsets $\{T_n\}_{n=1}^N \subset [0,T]$. If the offsets T_n are chosen independently and uniformly at random from $[0,T]$ and if the sample values of f are quantized with a first order Sigma-Delta algorithm, then with high probability the quantization error $|f(t)-\widetilde{f}(t)|$ is at most of order $N^{-1}\log N$.

In memory of David Gottlieb–mentor and friend.

1. Introduction

One of the fundamental issues in modern electronics is the conversion between analog and digital signal representations. Shannon's classical sampling theorem is a tool for passing between bandlimited signals and their representations from uniform samples. We shall focus on the space B_σ of finite energy bandlimited signals whose Fourier transforms are supported on $[-\sigma,\sigma]$. More precisely, $B_\sigma = \{f \in L^2(\mathbb{R}) : \text{supp}(\widehat{f}) \subset [-\sigma,\sigma]\}$, where the Fourier transform $\widehat{f}$ is normalized as $\widehat{f}(\xi) = \int f(t)e^{-2\pi i\xi t}dt$. Each $f \in B_\sigma$ has the representation

$$f(t) = \int_{-\sigma}^{\sigma} \widehat{f}(\xi)e^{2\pi i\xi t}d\xi.$$

Shannon's sampling theorem, e.g., see [17], states that each $f \in B_\sigma$ can be expressed exactly in terms of its sample values $\{f(kT)\}_{k\in\mathbb{Z}}$ through

$$f(t) = T\sum_{k\in\mathbb{Z}} f(kT)\psi(t-kT), \tag{1.1}$$

provided that $(\sigma+\sigma_c) \leq 1/T$, where the sampling kernel ψ satisfies

$$\widehat{\psi}(\xi) = \begin{cases} 1, & \text{when } |\xi| \leq \sigma, \\ 0, & \text{when } |\xi| > \sigma_c, \end{cases} \tag{1.2}$$

with $\sigma \leq \sigma_c$. In particular, it is necessary that $1/T \geq 2\sigma$.

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When the sampling rate $1/T$ is strictly greater than the Nyquist rate 2σ, i.e., $1/T > 2\sigma$, it is possible to take $\sigma_c > \sigma$ in (1.2) and to select $\widehat{\psi}$ to be smooth. This is an essential property for the usefulness of Shannon's sampling theorem in applications since in practice only a finite number of samples $f(kT)$ are available for the reconstruction of $f(t)$. As a result, the sum in (1.1) must be truncated and $f(t)$ is only approximately recovered. If $\widehat{\psi}$ is selected to be sufficiently smooth, e.g., at least twice continuously differentiable, then $\psi(t)$ will decay at least as rapidly as $|t|^{-2}$, and (1.1) can be truncated to a finite sum while still recovering accurate approximations of $f(t)$ in the sampled region, e.g., see [13]. Examples of practical kernels $\widehat{\psi}$ are plentiful, see [6, 10, 8, 11]; they range from the classical raised cosine to infinitely differentiable Gevrey-α filters [13]. In each case, $\widehat{\psi}(\xi)$ is defined by selecting a function $\rho(\xi)$ and letting

$$\widehat{\psi}_\rho(|\xi|) = \rho\left(\frac{|\xi| - \sigma}{\sigma_c - \sigma}\right) \quad \text{for} \quad \sigma \leq |\xi| \leq \sigma_c. \tag{1.3}$$

For the results presented here the classical raised-cosine defined by

$$\rho_{rc}(\xi) = \frac{1}{2}(1 + \cos(\pi\xi)). \tag{1.4}$$

yields sufficient localization, and we may for simplicity restrict our attention to the associated sampling kernel $\psi = \psi_{\rho_{rc}}$.

A practical limitation in the direct application of Shannon's sampling theorem is the increasingly high sampling rates that are necessary for ultra-wideband transmissions. The high rate analog-to-digital converters (ADCs) needed for such applications pose serious challenges such as decreased accuracy and increased cost. A naive solution for synthetically increasing the sampling rate is to interleave multiple analog-to-digital converters, each of which has a sampling rate $1/T$ that lies *below* the Nyquist rate. Interleaving N such ADCs to a uniform mesh gives a faster effective sampling rate of N/T. Unfortunately this naive approach is not easy to implement, especially for large N, due to the difficulty in synchronizing (and maintaining synchronization of) the interleaved ADCs [15]. This obstacle can be overcome by a suitable modification of Shannon's sampling theorem to allow for unsynchronized uniform interleaving, often called uniform interleaving [13, 14], e.g., see Theorem 2.2.

Overview and main results. The main contribution of this article is to give an analysis of the compatibility of uniform interleaved sampling with coarse quantization. We focus on coarse quantization given by Sigma-Delta ($\Sigma\Delta$) algorithms, and consider an unsynchronized sampling geometry with randomly interleaved sampling grids. It is desirable to use $\Sigma\Delta$ schemes in this setting because of their superior robustness properties: $\Sigma\Delta$ methods can be implemented using imperfect circuit elements without compromising the accuracy of the recovered approximation.

Our main result may be summarized as follows. Suppose that the bandlimited signal $f \in B_\sigma$ is sampled on an interleaved collection of N uniform grids $\{kT + T_n\}_{k\in\mathbb{Z}}$ with offsets $\{T_n\}_{n=1}^N$ that are chosen independently and uniformly at random from $[0, T]$. If the samples of f are quantized with a first order Sigma-Delta analog-to-digital converter then with quantifiably high probability the overall quantization error $\|f - \widetilde{f}\|_{L^\infty(\mathbb{R})}$ is at most of order $N^{-1}\log N$. This is within a logarithmic factor of the rate achieved with traditional uniform sampling. A precise statement of this main result is given in Theorem 4.2.

The remainder of the paper is organized as follows. Section 2 reviews unsynchronized uniform interleaved sampling of bandlimited signals. Section 3 focuses on the special case of randomly interleaved sampling, and derives size and variation estimates associated with the sampling kernels in this setting. Section 4.1 provides basic background on Sigma-Delta ($\Sigma\Delta$) quantizers and derives general error bounds in the setting of unsynchronized uniform interleaved sampling. Section 4.2 contains our main result, Theorem 4.2, which proves error bounds for Sigma-Delta quantization of randomly interleaved uniform sampling. Numerical examples are given in Section 5.

2. Uniform interleaved sampling of bandlimited signals

Similar to Shannon's sampling theorem (1.1) for uniform samples, a bandlimited function $f \in B_\sigma$ can also be expressed in terms of its uniform interleaved samples, i.e., samples on a union of shifted lattices. Each $f \in B_\sigma$ can be represented in terms of the interleaved uniform samples $\{f(kT+T_n) : k \in \mathbb{Z}, n = 1, \dots N\}$, where $\{T_n\}_{n=1}^N \subset \mathbb{R}$, through

$$f(t) = T \sum_{k\in\mathbb{Z}} \sum_{n=1}^{N} f(kT+T_n)\psi_n(t-kT-T_n),$$

provided that

- the sampling grids $\{kT+T_n\}_{k\in\mathbb{Z}}$ are disjoint for different $n = 1, 2, \cdots, N$,
- the effective sampling rate N/T is greater than the Nyquist rate 2σ,
- the sampling kernels $\{\psi_n\}_{n=1}^N$ are appropriately constructed.

See [13] for further details. In the case where N is appropriately large, it is possible to select each ψ_n to differ only by a constant multiple c_n from a standard sampling kernel ψ that satisfies (1.2), i.e., we may take $\psi_n = c_n\psi$.

We shall make use of the following standard result which is essentially a version of the Poisson summation formula. We include a proof for the sake of completeness.

Lemma 2.1. *If $\varphi \in B_\sigma$ and $\lambda > 0$ then*

$$\sum_{k\in\mathbb{Z}} \varphi(k/\lambda)e^{-2\pi i\xi k/\lambda} = \lambda \sum_{n\in\mathbb{Z}} \widehat{\varphi}(\xi+n\lambda). \tag{2.1}$$

The left sum in (2.1) converges unconditionally in $L^2_{loc}(\mathbb{R})$. The right sum in (2.1) locally involves only finitely many indices.

Proof. Since both sides of (2.1) are λ periodic, it suffices to verify that equality holds when both sides of (2.1) are restricted to $[0,\lambda]$. Let

$$s(\xi) = \chi_{[0,\lambda]}(\xi) \sum_{n\in\mathbb{Z}} \widehat{\varphi}(\xi+n\lambda). \tag{2.2}$$

Since $\widehat{\varphi} \in L^2(\mathbb{R})$ is compactly supported on $[-\sigma,\sigma]$, the sum defining s is only taken over finitely many indices. Hence $s \in L^2[0,\lambda]$.

Let m_1 be the largest integer such that $-\sigma - m_1\lambda \leq \lambda$, and let m_2 be the smallest integer such that $\sigma - m_2\lambda \geq 0$. Using the support properties of $\widehat{\varphi}$ we have that

$$\int_0^\lambda s(\xi)e^{2\pi ik\xi/\lambda}d\xi = \sum_{n=m_1}^{m_2}\int_0^\lambda \widehat{\varphi}(\xi+n\lambda)e^{2\pi ik\xi/\lambda}d\xi = \sum_{n=m_1}^{m_2}\int_{n\lambda}^{(n+1)\lambda}\widehat{\varphi}(\xi)e^{2\pi ik\xi/\lambda}d\xi$$
$$= \int_{m_1\lambda}^{(m_2+1)\lambda}\widehat{\varphi}(\xi)e^{2\pi ik\xi/\lambda}d\xi = \int_{-\sigma}^{\sigma}\widehat{\varphi}(\xi)e^{2\pi ik\xi/\lambda}d\xi = \varphi(k/\lambda).$$

Thus s has the Fourier series representation

$$s(\xi) = \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\varphi(k/\lambda)e^{-2\pi ik\xi/\lambda},$$

which converges unconditionally in $L^2[0,\lambda]$. This completes the proof. □

Theorem 2.2. *Suppose that $f \in B_\sigma$ is sampled on the set*

$$\{kT + T_n : k \in \mathbb{Z}, 1 \leq n \leq N\},$$

where $0 \leq T_1 < T_2 < \cdots < T_N < T$. Let ψ be a sampling kernel that satisfies (1.2) with associated bandwidth constant $\sigma_c \geq \sigma$. Let $m = \lceil T(\sigma+\sigma_c)\rceil$ and assume that the number of interleaved grids satisfies $N \geq 2m+1$.

Let the vector $c = [c_1, \cdots, c_N]^T$ be chosen as any solution to $Ac = e$, where the $(2m+1)\times N$ matrix A is defined by

$$\forall\ 1 \leq j \leq 2m+1 \ \ \textit{and}\ \ 1 \leq k \leq N, \quad A_{j,k} = e^{-2\pi i(j-m-1)T_k/T}, \tag{2.3}$$

and where the $(2m+1)\times 1$ vector $e = [e(1), \cdots, e(2m+1)]^T$ is defined by $e(n) = \delta_{n,(m+1)}$. The Kronecker $\delta_{j,k}$ is defined to equal one when $j = k$, and zero otherwise.

Then the following sampling formula holds

$$f(t) = T\sum_{k\in\mathbb{Z}}\sum_{n=1}^{N} f(kT+T_n)c_n\psi(t-kT-T_n), \tag{2.4}$$

with unconditional convergence in both $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$.

Proof. Define

$$f_n(t) = Tc_n\sum_{k\in\mathbb{Z}} f(kT+T_n)\psi(t-kT-T_n),$$

and note that $f_n \in L^2(\mathbb{R})$. The Fourier transform of f_n is given by

$$\widehat{f_n}(\xi) = Tc_n\widehat{\psi}(\xi)e^{-2\pi iT_n\xi}\sum_{k\in\mathbb{Z}} f(kT+T_n)e^{-2\pi ikT\xi}. \tag{2.5}$$

Note that $f_n \in L^2(\mathbb{R})$ and $\widehat{f_n}$ is supported on $[-\sigma_c, \sigma_c]$, because $\{f(kT+T_n)\}_{k\in\mathbb{Z}} \in \ell^2$ for every T_n and $\widehat{\psi}(\xi)$ is constructed to be compactly supported on $[-\sigma_c, \sigma_c]$.

Applying Lemma 2.1 to (2.5) with $\varphi(t) = f(t+T_n)$ and $\lambda = 1/T$ gives

$$\widehat{f_n}(\xi) = \widehat{\psi}(\xi)c_n\sum_{k\in\mathbb{Z}} e^{-2\pi ikT_n/T}\widehat{f}(\xi+k/T), \tag{2.6}$$

with equality in $L^2(\mathbb{R})$. Note that the sum is a finite sum since $f \in B_\sigma$.

For (2.4) to hold we require $f = \sum_{n=1}^N f_n$, or equivalently $\widehat{f} = \sum_{n=1}^N \widehat{f_n}$. By (2.6) we have

$$\sum_{n=1}^N \widehat{f_n}(\xi) = \widehat{\psi}(\xi) \sum_{k\in\mathbb{Z}} \widehat{f}(\xi + k/T) \sum_{n=1}^N c_n e^{-2\pi i k T_n/T}. \tag{2.7}$$

Note that a circular shift of the rows of A results in a Vandermonde matrix, and consequently A has full rank if $0 \le T_1 < T_2 < \cdots < T_N < T$. Thus by the definition of c as a solution to $Ac = e$ we have that

$$\forall\ |k| \le m, \quad \sum_{n=1}^N c_n e^{-2\pi i k T_n/T} = \delta_{k,0}. \tag{2.8}$$

Also, by the definition of m, and since $\mathrm{supp}(\widehat{\psi}) \subset [-\sigma_c, \sigma_c]$ and $\mathrm{supp}(\widehat{f}) \subset [-\sigma, \sigma]$, we have that

$$\forall\ |k| > m, \quad \widehat{\psi}(\xi)\widehat{f}(\xi + k/T) = 0. \tag{2.9}$$

Combining (2.7), (2.9), and (2.8), gives $\sum_{n=1}^N \widehat{f_n}(\xi) = \widehat{f}(\xi)$, as required. This shows that (2.4) holds in $L^2(\mathbb{R})$.

To see that (2.4) also holds in $L^\infty(\mathbb{R})$ recall that $f \in B_\sigma \subset B_{\sigma_c}$ and note that each function in the right side of (2.4) is in B_{σ_c}. Thus the L^2 convergence of the respective Fourier transforms in fact takes place in $L^2[-\sigma_c, \sigma_c]$. Since

$$\forall\ h \in B_\sigma, \quad \|h\|_{L^\infty(\mathbb{R})} \le \|\widehat{h}\|_{L^1(\mathbb{R})} \le \sqrt{2\sigma_c}\|\widehat{h}\|_{L^2(\mathbb{R})},$$

the convergence in $L^\infty(\mathbb{R})$ follows. □

For perspective on the effective sampling rate in Theorem 2.2, note that the hypotheses on m and N imply that $N/T > 2\sigma$. It is also worth mentioning that Theorem 2.2 offers many degrees of design flexibility. Besides the freedom of choosing ψ, the system $Ac = e$ becomes increasingly underdetermined as $N \to \infty$ (with σ, σ_c, T fixed) and there are infinitely many choices for the coefficients $\{c_n\}_{n=1}^N$.

3. Random uniform interleaved sampling of bandlimited signals

In this section we consider the case of unsynchronized interleaved sampling with no deterministic control on the offsets $0 \le T_1 < T_2 < \cdots < T_N < T$ from Theorem 2.2. We consider the situation where $\{T_n\}_{n=1}^N$ is determined by choosing N independent uniform random variables on $[0, T]$ and then arranging them in increasing order. In other words, $\{T_n\}_{n=1}^N$ will be assumed to be the order statistics associated with N independent random variables each uniformly distributed on $[0, T]$.

There are powerful technical results, notably in [1, 9], which justify the effectiveness of randomly interleaved sampling. Since these results focus primarily on sampling theoretic issues, we hope that the current work on coarse analog-to-digital conversion will serve as a useful complement. The results and techniques in [1, 9] play an important role in our analysis and motivation.

In order to show that this unsynchronized sampling architecture is compatible with coarse quantization algorithms, we need to carry out a refined investigation of the coefficients $\{c_n\}_{n=1}^N$ that arise in Theorem 2.2. This section will focus on proving size and smoothness bounds for the canonical choice of the sequence $\{c_n\}_{n=1}^N$. We begin by stating some background results that will be important for our analysis.

Theorem 3.1, which is a streamlined corollary of [9, Theorem 3.3], bounds the smallest singular value of the matrix A when the sampling offsets $\{T_n\}_{n=1}^N$ are the order statistics associated with N independent random variables each uniformly distributed on $[0,T]$, see also [1]. For further details on the following theorem, see [9].

Theorem 3.1. *Let $\{T_n\}_{n=1}^N$ be defined by drawing N independent uniform random variables on $[0,T]$ and arranging them increasing order. Let A be the $(2m+1)\times N$ matrix defined by (2.3) with $N > (2m+1)$. Then with probability at least*

$$1 - 4(m+1)e^{-N/(40(m+1))},$$

the following upper bound on the operator norm of $(AA^*)^{-1}$ is satisfied

$$\|(AA^*)^{-1}\|_2 \leq 20N^{-1}. \tag{3.1}$$

Note that the original theorem in [9] is stated for $\{T_n\}_{n=1}^N$ chosen as independent uniform random variables on $[0,T]$. Since the singular values of a matrix is not affected by any interchange of its columns, the same result holds when the $\{T_n\}_{n=1}^N$ are the associated order statistics.

The next theorem bounds the maximal gap in a collection of independent uniform random variables.

Theorem 3.2. *Let $\{T_n\}_{n=1}^N$ be defined by drawing N independent uniform random variables on $[0,T]$ and arranging them increasing order, and consider the associated maximal gap defined by*

$$M_N = \max\{T_2 - T_1, \cdots, T_n - T_{n-1}, \cdots, T_N - T_{N-1}, T + T_1 - T_N\}.$$

Fix $\alpha > 0$. If $N \geq 4(1+\alpha)^2$ then with probability at least

$$1 - 2/N^\alpha$$

there holds

$$M_N \leq \frac{(1+\alpha)T\log N}{N}.$$

Proof. Without loss of generality, we consider the case $T = 1$, since the case of $T \neq 1$ then follows by rescaling.

For independent uniform random variables on $[0,1]$, the distribution for the difference $T_k - T_j$ of a pair of order statistics with $j < k$ is given by

$$f_{(T_k - T_j)}(x) = \begin{cases} \frac{N!\, x^{k-j-1}(1-x)^{N-k+j}}{(k-j-1)!(N-k+j)!}, & \text{if } x \in [0,1], \\ 0, & \text{if } x \notin [0,1]. \end{cases}$$

See equation (2.3.4) on page 14 in [4] for further details. For differences of adjacent order statistics and $x \in [0,1]$ this reduces to

$$\forall\ 1 \leq n \leq N-1, \quad f_{(T_{n+1} - T_n)}(x) = N(1-x)^{N-1},$$

and

$$f_{(T_N - T_1)}(x) = N(N-1)x^{N-2}(1-x).$$

Fix $0 < \lambda < 1$. For any fixed n we have that

$$\text{Prob}[T_{n+1} - T_n > \lambda] = N\int_\lambda^1 (1-x)^{N-1}dx = (1-\lambda)^N. \tag{3.2}$$

Similarly,

$$\begin{aligned}\text{Prob}[1+T_1-T_N>\lambda] &= \text{Prob}[1-\lambda>T_N-T_1]\\ &= N(N-1)\int_0^{1-\lambda} x^{N-2}(1-x)dx\\ &= N\lambda(1-\lambda)^{N-1}+(1-\lambda)^N. \end{aligned}\tag{3.3}$$

The probability that at least one of the differences $\{T_{n+1}-T_n\}_{n=1}^{N-1}$ or $1+T_1-T_N$ exceeds λ can be bounded using a union bound with (3.2) and (3.3) to give

$$\begin{aligned}\text{Prob}[M_N>\lambda] &\leq (N-1)(1-\lambda)^N + N\lambda(1-\lambda)^{N-1}+(1-\lambda)^N\\ &= N(1-\lambda)^N\left(\frac{1}{1-\lambda}\right)\\ &\leq Ne^{-\lambda N}\left(\frac{1}{1-\lambda}\right). \end{aligned}\tag{3.4}$$

To complete the proof we select $\lambda=\lambda_N=(1+\alpha)N^{-1}\log N$ and note that for $N>4(1+\alpha)^2$ we have $0<\lambda<1/2$ and $(1-\lambda)^{-1}<2$. Thus, by (3.4), $N>4(1+\alpha)^2$ implies $\text{Prob}[M_N>\lambda]\leq 2Ne^{-\lambda N}$, and hence by the definition of λ

$$\text{Prob}\left[M_N>\frac{(1+\alpha)\log N}{N}\right]\leq\frac{2}{N^\alpha}.$$

□

Although the bound in Theorem 3.2 suffices for our purposes, there are also more delicate asymptotic bounds in [5], cf. [12], which show that with probability one,

$$\limsup_{N\to\infty}\frac{NM_N-\log N}{2\log\log N}=1.$$

Also see [1] for a version in higher dimensions.

The following theorem catalogs properties of the vector c from Theorem 2.2 in the setting of $\{T_n\}_{n=1}^N$ drawn uniformly at random from $[0,T]$. Given $\{c_n\}_{n=1}^N\subset\mathbb{C}^N$, we define $\Delta c_n=c_n-c_{n+1}$ if $1\leq n\leq N-1$, and $\Delta c_N=c_N-c_1$. In the interest of concreteness we state the following theorem for specific choices of the bounding constants and associated probabilities. More general versions follow in the same manner using a more general version of Theorem 3.1 as stated in [9, Theorem 3.3].

Theorem 3.3. *Let $\{T_n\}_{n=1}^N$ be defined by drawing N independent uniform random variables on $[0,T]$ and arranging them increasing order. Let A be the $(2m+1)\times N$ matrix defined by (2.3) with $N>2m+1$, let $e=[e_1,\cdots,e_{2m+1}]^T$ be defined by $e_n=\delta_{n,(m+1)}$, and let $c=[c_1,\cdots,c_N]^T$ denote the canonical solution to $Ac=e$ that is given by*

$$c=A^*(AA^*)^{-1}e.$$

Fix $\alpha>0$ and additionally assume $N>4(1+\alpha)^2$. Then with probability at least

$$1-\frac{2}{N^\alpha}-4(m+1)e^{-N/(40(m+1))}$$

the following hold:

$$\max_{1\leq n\leq N}|c_n|\leq\frac{20\sqrt{2m+1}}{N},\tag{3.5}$$

and there exists a constant $0 < C_m \leq 2\pi(2m+1)^{3/2}$ *such that*

$$\max_n |\Delta c_n| \leq \frac{20(1+\alpha)C_m \log N}{N^2}. \tag{3.6}$$

Proof. We first establish (3.5). For a row or column vector v, we use $v(n)$ to denote the nth entry of v. Let

$$a_n = [e^{2\pi i m T_n/T}, e^{2\pi i(m-1)T_n/T}, \cdots, e^{-2\pi i m T_n/T}]^T$$

denote the nth column of the matrix A. Using this notation, and recalling that $e(n) = \delta_{n,(m+1)}$ we have that $c = A^*(AA^*)^{-1}e$ is the $(m+1)$th column of $A^*(AA^*)^{-1}$. So c^* is the $(m+1)$th row of the matrix $[(AA^*)^{-1}]^*A = (AA^*)^{-1}A$ and hence is given by

$$c^*(n) = \overline{c(n)} = ((AA^*)^{-1}a_n)(m+1),$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Note that $\|a_n\|_2 = \sqrt{2m+1}$. Thus, by Theorem 3.1, with probability greater than $1 - 4(m+1)e^{-N/(40(m+1))}$

$$\begin{aligned} |c(n)| &= \left|((AA^*)^{-1}a_n)(m+1)\right| \\ &\leq \|(AA^*)^{-1}a_n\|_2 \\ &\leq \|(AA^*)^{-1}\| \; \|a_n\|_2 \\ &\leq 20N^{-1}\sqrt{2m+1}. \end{aligned} \tag{3.7}$$

To establish (3.6) we begin with the bound

$$\begin{aligned} |c(n) - c(n-1)| &= |\overline{c(n)} - \overline{c(n-1)}| \\ &= |((AA^*)^{-1}a_n)(m+1) - ((AA^*)^{-1}a_{n-1})(m+1)| \\ &= |((AA^*)^{-1}(a_n - a_{n-1}))(m+1)| \\ &\leq \|(AA^*)^{-1}(a_n - a_{n-1})\|_2 \\ &\leq \|(AA^*)^{-1}\| \; \|a_n - a_{n-1}\|_2 \end{aligned} \tag{3.8}$$

Again, Theorem 3.1 supplies a bound on $\|(AA^*)^{-1}\|$. We now address how to control $\|a_n - a_{n-1}\|_2$. Note that $a_n = g(T_n)$ where g is the smooth vector valued function

$$g(t) = [g_{-m}(t), g_{-m+1}(t), \ldots, g_m(t)]^T,$$

with

$$g_j(t) = \exp(2\pi i j t/T).$$

Observe that $|g_j'(t)| \leq 2\pi j/T$. Then, using the mean-value theorem on g_j, we get

$$|g_j(T_n) - g_j(T_{n-1})| \leq \frac{2\pi j}{T}|T_n - T_{n-1}|,$$

and consequently

$$\|a_n - a_{n-1}\|_2 = \|g(T_n) - g(T_{n-1})\|_2 \leq \frac{C_m}{T}|T_n - T_{n-1}|, \tag{3.9}$$

where

$$C_m = 2\pi(\sum_{j=-m}^{m} j^2)^{1/2} = 2\pi\sqrt{\frac{m(m+1)(2m+1)}{3}} \leq 2\pi(2m+1)^{3/2}.$$

Next, by Theorem 3.2, with probability greater than $1 - 2/N^\alpha$

$$|T_n - T_{n-1}| \leq \frac{(1+\alpha)T\log N}{N}. \tag{3.10}$$

Combining (3.10) with (3.9) gives

$$\|a_n - a_{n-1}\| \leq \frac{(1+\alpha)C_m \log N}{N}. \tag{3.11}$$

The bounds in (3.8) and (3.11) and Theorem 3.1 combine to give the desired result (3.6)

$$|c(n) - c(n-1)| \leq \frac{20(1+\alpha)C_m \log N}{N^2},$$

with the same bound holding analogously for $|c(N) - c(1)|$. □

4. Sigma-Delta quantization

4.1. **Sigma-Delta for general uniform interleaved sampling.** This section briefly introduces Sigma-Delta ($\Sigma\Delta$) quantization and states a general error bound that relates quantization error and variation of the sampling kernel.

Signal expansions such as (1.1) and (2.4), via appropriate sampling theorems, give discrete-time representations of bandlimited functions. But the sample values in such decompositions still take on a continuous range of real values. Quantization is the lossy process of discretizing sample amplitudes to lie in a finite set such as $\{-1, 1\}$. This makes it possible to provide representations that are discrete in both time and amplitude.

Suppose that $f \in B_\sigma$ is real valued, that $\{T_n\}_{n=1}^N \subset [0, T]$ is strictly increasing, and that the interleaved sampling formula (2.4) holds. We focus on the case where the samples are quantized in the time order in which they are obtained, so we order the sequence $\{f(kT + T_n) : k \in \mathbb{Z}, 1 \leq n \leq N\}$ according to the order imposed by sampling and quantize this sequence using a stable first order $\Sigma\Delta$ quantization scheme. Noting that

$$\cdots < T_N - T < T_1 < T_2 < \cdots < T_N < T_1 + T < \dots \tag{4.1}$$

we denote this time ordered sequence of sample instances by $\{t_m\}_{m\in\mathbb{Z}}$ with $t_1 = T_1$. In other words, $\{t_m\}_{m\in\mathbb{Z}}$ is defined by arranging $\{kT + T_n : k \in \mathbb{Z}, 1 \leq n \leq N\}$ in increasing order.

The next definition uses the 1-bit scalar quantizer $Q : \mathbb{R} \to \{-1, 1\}$ given by

$$Q(u) = \begin{cases} 1, & \text{if } u \geq 0, \\ -1, & \text{if } u < 0. \end{cases} \tag{4.2}$$

There is no difficulty in extending our analysis to multibit quantizers, e.g., see [2], but we restrict our attention to the 1-bit case for the sake of simplicity.

Definition 4.1. The *first order* $\Sigma\Delta$ *scheme* is given by the following recursion. Let $u_0 = 0$.

$$\begin{aligned} \text{For } m \geq 1: \qquad q_m &= Q(u_{m-1} + f(t_m)), \\ u_m &= u_{m-1} + f(t_m) - q_m, \\ \text{for } m < 1: \qquad q_m &= -Q(u_m - f(t_m)), \\ u_{m-1} &= u_m - f(t_m) + q_m. \end{aligned}$$

The $\Sigma\Delta$ algorithm takes the sequence of sample values $\{f(t_m)\}_{m\in\mathbb{Z}}$ as its input and returns the quantized sequence $\{q_m\}_{m\in\mathbb{Z}} \subset \{-1, 1\}$ as it output. The auxiliary sequence $\{u_m\}_{m\in\mathbb{Z}}$ is referred to as the *state variable* sequence. The $\Sigma\Delta$ algorithm is *stable* in the following sense. If the input sample sequence satisfies $|f(t_m)| < 1$

for all $m \in \mathbb{Z}$, then it is well known that the state variables satisfy $|u_m| \leq 1$, see [3]. In practice one runs the $\Sigma\Delta$ recursion only for finitely many positive indices m since $m = 0$ corresponds to the time index when the sampling operation begins.

Below, we give a generic upper bound on the $\Sigma\Delta$ quantization error.

Proposition 4.1. *Let $\{t_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ be any increasing sequence. Suppose that f admits a decomposition*

$$f(t) = \sum_{m\in\mathbb{Z}} f(t_m)\psi_m(t - t_m)$$

where each ψ_m vanishes at infinity, i.e., $\lim_{|t|\to\infty} \psi_m(t) = 0$. Let $\{q_m\}_{m\in\mathbb{Z}}$ be the sequence obtained by quantizing the sample sequence $\{f(t_m)\}_{m\in\mathbb{Z}}$ using the (two-sided) stable first order $\Sigma\Delta$ scheme. Then

$$|f(t) - \sum_{m\in\mathbb{Z}} q_m\psi_m(t - t_m)| \leq \sum_{m\in\mathbb{Z}} |\psi_m(t - t_m) - \psi_{m+1}(t - t_{m+1})|. \tag{4.3}$$

Proof. Note that

$$\begin{aligned} f(t) - \sum_{m\in\mathbb{Z}} q_m\psi_m(t - t_m) &= \sum_{m\in\mathbb{Z}} (f(t_m) - q_m)\psi_m(t - t_m) \\ &= \sum_{m\in\mathbb{Z}} (u_m - u_{m-1})\psi_m(t - t_m) \\ &= \sum_{m\in\mathbb{Z}} u_m \left[\psi_m(t - t_m) - \psi_{m+1}(t - t_{m+1})\right], \end{aligned}$$

where the last equality is obtained using summation by parts and noticing that the boundary terms disappear for all t since ψ_m vanishes at infinity. Finally, using the fact that $|u_m| < 1$, we obtain (4.3). □

Next, we focus on the sampling sequences and reconstruction kernels that are relevant in the setting of interleaved sampling. In the following theorem $\{t_m\}_{m\in\mathbb{Z}}$ denotes the ordered sampling instances $\{Tk + T_n : k \in \mathbb{Z}, 1 \leq n \leq N\}$ as in (4.1).

Proposition 4.2. *Let $f \in B_\sigma$ be real valued and satisfy $\|f\|_{L^\infty(\mathbb{R})} < 1$. Suppose that f is sampled on the interleaved grids $\{Tk + T_n : k \in \mathbb{Z}, 1 \leq n \leq N\}$ as in Theorem 2.2, so that the sampling expansion (2.4) holds.*

Quantize the time ordered samples $\{f(t_m)\}_{m\in\mathbb{Z}}$ with the first order $\Sigma\Delta$ scheme to give the quantized samples $\{q_m\}_{m\in\mathbb{Z}}$ and denote the approximation $\widetilde{f}$ obtained using the quantized samples by

$$\widetilde{f}(t) = \sum_{m\in\mathbb{Z}} q_m c_m \psi(t - t_m), \tag{4.4}$$

where the constants c_m are as in Theorem 2.2. Then the following quantization error bound holds

$$|f(t) - \widetilde{f}(t)| \leq 2K_1(N)\|\psi\|_{L^1(\mathbb{R})} + K_2(N)N\big(T^{-1}\|\psi\|_{L^1(\mathbb{R})} + \|\psi'\|_{L^1(\mathbb{R})}\big), \tag{4.5}$$

where

$$K_1(N) = \max_{1\leq n\leq N} |c_n| \quad \text{and} \quad K_2(N) = \max_{1\leq n\leq N} |\Delta c_n|.$$

Proof. Recalling that $\{t_m\}_{m\in\mathbb{Z}}$ is the time ordered version of $\{Tk+T_n : k\in\mathbb{Z}, 1\leq n\leq N\}$, it will be convenient to introduce the notation $T_{n,k}=Tk+T_n$. Using Proposition 4.1 one can verify that

$$\begin{aligned}
|f(t)-\widetilde{f}(t)| &\leq \sum_{k\in\mathbb{Z}}\sum_{n=1}^{N-1}|c_{n+1}||\psi(t-T_{n+1,k})-\psi(t-T_{n,k})| \\
&+ \sum_{k\in\mathbb{Z}}\sum_{n=1}^{N-1}|c_{n+1}-c_n||\psi(t-T_{n,k})| \\
&+ \sum_{k\in\mathbb{Z}}|c_N||\psi(t-T_{N,k})-\psi(t-T_{1,k+1})| \\
&+ \sum_{k\in\mathbb{Z}}|c_1-c_N||\psi(t-T_{1,k+1})|.
\end{aligned}$$

Denote the four sums above by $I_{1,1}$, $I_{1,2}$, $I_{2,1}$, and $I_{2,2}$ respectively. Next, we find upper bounds for each sum.

(i) First, we bound $I_{1,1}$. For the sufficiently smooth sampling kernels considered in this article we have $\psi'\in L^1(\mathbb{R})$, and hence

$$\begin{aligned}
|I_{1,1}(t)| &\leq K_1(N)\sum_{k\in\mathbb{Z}}\sum_{n=1}^{N-1}\Big|\int_{t-T_{n+1}-kT}^{t-T_n-kT}\psi'(\tau)d\tau\Big| \\
&\leq K_1(N)\sum_{k\in\mathbb{Z}}\int_{t-(k+1)T}^{t-kT}|\psi'(\tau)|d\tau \\
&\leq K_1(N)\|\psi'\|_{L^1(\mathbb{R})} \qquad (4.6)
\end{aligned}$$

where $K_1(N)=\max_{1\leq n\leq N}|c_n|$. In the second inequality, we used the fact that $\{T_n\}_{n=1}^N$ is strictly increasing in $[0,T)$.

(ii) Next, we bound $I_{1,2}$. Setting $K_2(N)=\max_{1\leq n\leq N}|\Delta c_n|$,

$$|I_{1,2}(t)|\leq K_2(N)\sum_{n=1}^{N-1}\sum_{k\in\mathbb{Z}}|\psi(t-T_n-kT)|. \qquad (4.7)$$

It may be verified that for any function $\psi\in L^1(\mathbb{R})\cap C^1(\mathbb{R})$ with $\psi'\in L^1(\mathbb{R})$, we have

$$\forall t,\quad \sum_{k\in\mathbb{Z}}|\psi(t-kT)|\leq T^{-1}\|\psi\|_{L^1(\mathbb{R})}+\|\psi'\|_{L^1(\mathbb{R})}. \qquad (4.8)$$

Consequently, (4.7) implies

$$|I_{1,2}(t)|\leq K_2(N)(N-1)(T^{-1}\|\psi\|_{L^1(\mathbb{R})}+\|\psi'\|_{L^1(\mathbb{R})}).$$

(iii) We have

$$\begin{aligned}
|I_{2,1}(t)| &\leq K_1(N)\sum_{k\in\mathbb{Z}}|\psi(t-T_N-kT)-\psi(t-T_1-(k+1)T)| \\
&\leq K_1(N)\sum_{k\in\mathbb{Z}}\int_{t-T_1-(k+1)T}^{t-T_N-kT}|\psi'(\tau)|d\tau \\
&\leq K_1(N)\|\psi'\|_{L^1(\mathbb{R})}. \qquad (4.9)
\end{aligned}$$

In the last inequality, we used the fact that $0 \le T_1 < T_N < T$, and consequently, the sets $U_k(t) = [t - T_1 - (k+1)T, t - T_N - kT]$ satisfy $U_k(t) \cap U_\ell(t) = \emptyset$ for all t whenever $k \neq \ell$.

(iv) Finally,

$$\begin{aligned} |I_{2,2}(t)| &\le |c_1 - c_N| \sum_{k\in\mathbb{Z}} |\psi(t - T_1 - (k+1)T)| \\ &\le K_2(N)(T^{-1}\|\psi\|_{L^1(\mathbb{R})} + \|\psi'\|_{L^1(\mathbb{R})}). \end{aligned}$$

Above we use (4.8) to obtain the second inequality. □

4.2. **Sigma-Delta for random uniform interleaved sampling.** We are now ready to state our main result. Theorem 4.2 addresses the behavior of first order Sigma-Delta quantization for unsynchronized interleaved sampling when the sampling offsets $\{T_n\}_{n=1}^N$ are chosen randomly on $[0, T]$.

Theorem 4.2. *Let $\{T_n\}_{n=1}^N$ be defined by drawing $N > 2m+1$ independent uniform random variables on $[0, T]$ and arranging them in increasing order. Suppose that a real valued signal $f \in B_\sigma$ satisfying $\|f\|_{L^\infty(\mathbb{R})} < 1$ is sampled on the interleaved collection of grids*

$$\{kT + T_n : k \in \mathbb{Z}, 1 \le n \le N\},$$

and that the corresponding time ordered samples $\{f(t_k)\}_{k\in\mathbb{Z}}$ are quantized with the first order $\Sigma\Delta$ scheme as in (4.1) and Definition 4.1 to give the quantized signal $\widetilde{f}$ in (4.4). Then, with probability greater than

$$1 - 2N^{-1} - 4(m+1)e^{-N/(40(m+1))},$$

there holds

$$|f(t) - \widetilde{f}(t)| \le \frac{C_1 + C_2 \log N}{N},$$

with $C_1 = 40\sqrt{2m+1}\|\psi\|_{L^1(\mathbb{R})}$ and $C_2 = 80\pi(2m+1)^{3/2}\big(T^{-1}\|\psi\|_{L^1(\mathbb{R})} + \|\psi'\|_{L^1(\mathbb{R})}\big)$.

Proof. First note that with probability one there holds

$$0 < T_1 < T_2 < \cdots < T_N < 1,$$

so that Theorem 2.2 applies and the sampling expansion (2.4) holds. Taking $\alpha = 1$ in Theorem 3.3 shows that with probability at least $1 - 2N^{-1} - 4(m+1)e^{-N/(40(m+1))}$ both (3.5) and (3.6) hold. Thus for $K_1(N)$ and $K_2(N)$ as in Proposition 4.2, we have

$$K_1(N) \le \frac{20\sqrt{2m+1}}{N} \quad \text{and } K_2(N) \le \frac{80\pi(2m+1)^{3/2}\log N}{N^2}.$$

Substituting this into (4.5) yields the desired result. □

It is well known that $\Sigma\Delta$ schemes have superior robustness properties with respect to quantizer imperfections. Theorem 4.2 uses the first order $\Sigma\Delta$ scheme given in Definition 4.1. The 1-bit quantizer Q of this scheme is an ideal comparator which is difficult to implement in practice. A more realistic model for the quantizer can be obtained by replacing $Q(\cdot)$ in Definition 4.1 with $Q(\cdot + \epsilon_n)$ where ϵ_n is unknown and may change at every step of the iteration, but can be kept within some known margin, i.e., $|\epsilon_n| < \epsilon$. In this case, the results shown in Theorem 4.2 remain valid after introducing the extra multiplicative constant $(1 + \epsilon)$, e.g., [3].

The $N^{-1}\log N$ error bound of Theorem 4.2 is consistent with similar error bounds for $\Sigma\Delta$ quantization in other settings. For example, a first order $\Sigma\Delta$ scheme produces an approximation with the error behaving like $1/\lambda$ where λ is the sampling rate in the settings of:

- regularly oversampled bandlimited signals in B_σ, [3],
- redundant finite frame expansions in $\mathbb{R}^d$, [2],
- overcomplete Gabor frames in $L^2(\mathbb{R})$, [16].

In this paper, our sampling rate is proportional to N, but the upper bounds given in Theorem 4.2 have an extra logarithmic factor $\log N$. The $\log N$ term is a consequence of the random (non)synchronization imposed by our sampling geometry.

It may be possible to improve the $N^{-1}\log N$ bound in Theorem 4.2 using more refined analysis. For example, if one takes subtle distribution properties of the state variables u_n into account, then it is possible to improve error bounds below the basic $1/\lambda$ bound in the settings of: regularly sampled bandlimited signals [7], and finite frames [2]. Indeed, in our setting of randomly interleaved sampling, numerical experiments indicate better performance than the $N^{-1}\log N$ bound, which leads us to believe that Theorem 4.2 can be improved.

5. Numerical examples

In this section we numerically illustrate the convergence rate of first order Sigma-Delta quantization applied to random uniform interleaved sampling of bandlimited signals, see Theorem 4.2. The following example uses specific choices of parameters for the sampling geometry and associated kernels: $\sigma = 1$, $\sigma_c = 1.2$, $T = 5$, $m = 11$.

Example 5.1. A real valued test signal $f \in B_\sigma$ satisfying $\sigma = 1$ and $\|f\|_{L^\infty(\mathbb{R})} < 1$ is selected. The test signal f used throughout this example is shown in Panel (a) of Figure 5.1. For different values of N, the signal f is sampled on a truncated portion of N interleaved sub-Nyquist grids

$$\{kT + T_n : k \in \mathbb{Z}, 1 \le n \le N\}.$$

In this example, we take $T = 5$ and the offsets $\{T_n\}_{n=1}^N$ are chosen by drawing N independent uniform random variables from $[0, T]$ and arranging them in increasing order. Truncation in the experiment only keeps 11 samples from each grid as follows

$$\{kT + T_n : -5 \le k \le 5, 1 \le n \le N\}. \tag{5.1}$$

As described in Section 4, the time ordered samples of f on the truncated grids (5.1) are quantized with the first order $\Sigma\Delta$ scheme. The signal $\widetilde{f}$ is reconstructed from the quantized samples as in (4.4), where ψ is the raised cosine filter (1.4) with $\sigma_c = 1.2$, and $\{c_n\}_{n=1}^N$ is taken as in Theorem 2.2 with $m = 11$. A zoomed-in comparison of $f(t)$ and $\widetilde{f}(t)$ is shown in Panel (b) of Figure 5.1 for $N = 6400$.

Table 1 considers $\max_{|t|\le 10} |f(t) - \widetilde{f}(t)|$ for various choices of N, and also shows that the bounds of $\|(AA^*)^{-1}\|_2$ and M_N in Theorems 3.1 and 3.2 respectively are satisfied. Figure 5.2 shows a log-log plot of the approximation error from Table 1 against N. The maximum error is considered for the subinterval $|t| \le 18$ to avoid the effect of truncation error; for further discussion on the role of truncation effects in sampling see [13].

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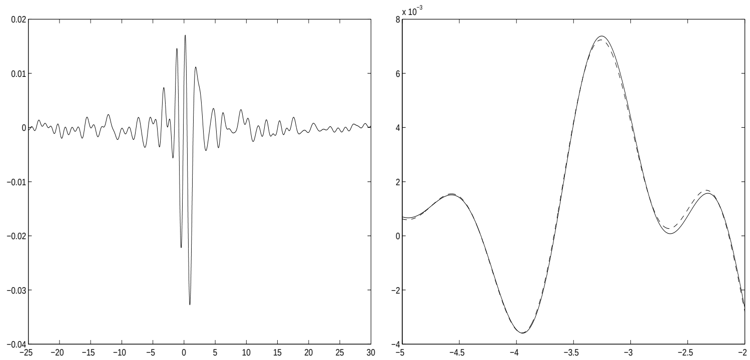

(a) (b)

FIGURE 5.1. Panel (a): Test signal $f \in B_1$ used in the numerical experiment. Panel (b): Zoomed-in view of the signal f (solid), and the reconstruction $\widetilde{f}$ (dashed) from first order $\Sigma\Delta$ quantization.

N	$\max_{\|t\|\leq 18} \|f(t) - \widetilde{f}(t)\|$	$N\\|(AA^*)^{-1}\\|_2$	M_N
50	4.1(-2)	48.0	5.65
100	7.0(-3)	4.68	5.43
200	3.4(-3)	3.03	5.74
400	6.7(-4)	1.84	6.17
800	2.4(-4)	1.40	4.51
1600	1.2(-4)	1.26	4.78
3200	4.6(-5)	1.20	4.70
6400	2.0(-5)	1.11	5.23

TABLE 1. For the signal displayed in Figure 5.1 we tabulate the: Maximal error, $\max_{|t|\leq 18} |f(t) - \widetilde{f}(t)|$ as well as observed values for $N\|(AA^*)^{-1}\|_2$ and M_N which are consistent with Theorems 3.1 and 3.2. The maximum error vs. N is shown in Figure 5.2.

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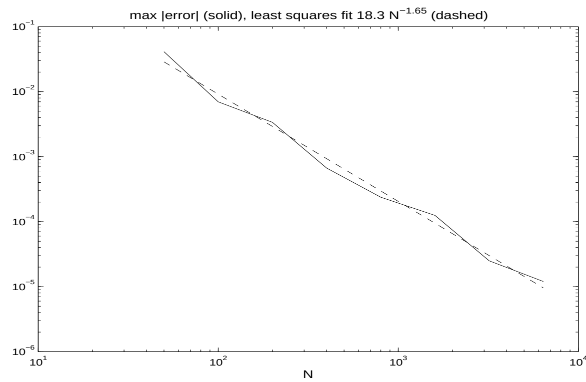

FIGURE 5.2. Log-log plot of the maximal error (solid) $\max_{|t|\leq 18}|f(t)-\widetilde{f}(t)|$. Least squares fit $10.2N^{-1.54}$ overlayed (dashed).

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