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MEAN SENSITIVE, MEAN EQUICONTINUOUS AND ALMOST PERIODIC FUNCTIONS FOR DYNAMICAL SYSTEMS

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Abstract. We show that a continuous abelian action (in particular $\mathbb{R}^d$) on a compact metric space equipped with an invariant ergodic measure has discrete spectrum if and only it is $\mu-$mean equicontinuous (proven for $\mathbb{Z}^d$ in [14]). In order to do this we introduce mean equicontinuity and mean sensitivity with respect to a function. We study this notion in the topological and measure theoretic setting. In the measure theoretic case we characterize almost periodic functions with these concepts and in the topological case we show that weakly almost periodic functions are mean equicontinuous (the converse does not hold). We compare our results with some results in the theory of Delone dynamical systems and quasicrystals.

A $\mathbb{Z}-$ topological dynamical system ($\mathbb{Z}-$TDS) is a pair (X,T) where X is a compact metric space (with metric d) and $T: X \to X$ a continuous invertible function (we will deal with abelian $\mathbb{G}$ actions, but for the introduction using $\mathbb{Z}$ will be enough to get intuition on the definitions). In the theory of topological dynamical systems several notions of order have been studied. The most ordered systems are the periodic systems, i.e. when for every $x \in X$ there exists $n_x > 0$ such that $T^{n_x}x = x$; followed by the equicontinuous systems i.e. when for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then $d(T^i x, T^i y) \leq \varepsilon$ for every $i \geq 0$. This notion means that the system if highly predictable in the following sense: if you only know x is inside a small $\delta-$neighbourhood you will be able to predict with $\varepsilon-$precision the orbit of x. Several weaker notions like distality, nullness, tameness, mean equicontinuity, among others have been studied. In particular we are interested in mean equicontinuity. A system is mean equicontinuous , if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then $\limsup \frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) \leq \varepsilon$. Here we have predictability not for every i but on sets of high density (see Proposition 1.23). According to Asulander [2], Fomin introduced these systems in [11].

Mean equicontinuity has received interest in recent years due to connections with ergodic properties of measurable dynamical systems, i.e. dynamical systems

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equipped with an invariant probability measure. In particular, it has been shown that using a measure theoretic version of mean equicontinuity one can characterize when a measure-preserving system has discrete spectrum [14] (i.e. when the eigenfunctions of the Koopman operator generate L^2) and when the maximal equicontinuous factor is actually an isomorphism (i.e. when the continuous eigenfunctions of the Koopman operator generate L^2) [22, 8] . The proof of the characterization of discrete spectrum using mean equicontinuity in [14] does not hold for continuous actions (e.g. $\mathbb{R}^d$). In this paper we prove that the characterization of discrete spectrum also holds for abelian $\mathbb{G}$-actions (Theorem 3.10). To do this we introduce mean equicontinuity with respect to a function. We say that (X,T) is mean equicontinuous with respect to $f: X \to \mathbb{C}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then $\limsup \frac{1}{n}\sum_{i=1}^{n} \left|f(T^i x) - f(T^i y)\right| \leq \varepsilon$. Actually we will see that a system is mean equicontinuous if and only if it is mean equicontinuous with respect to every continuous function. Using the measure theoretic version of this concept we can characterize almost periodic functions (Corollary 3.8). Almost periodic functions are classical objects in ergodic theory that are used to characterize discrete spectrum and weak mixing.

Another important concept that appears in the paper is sensitivity. A topological dynamical system is *sensitive* if there exists $\varepsilon > 0$ such that for every $x, y \in X$ there exists i such that $d(T^i x, T^i y) \geq \varepsilon$ and *mean sensitive* if there exists $\varepsilon > 0$ such that for every $x, y \in X$ we have $\limsup \frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) \geq \varepsilon$.

We study similar concepts in three different categories of dynamical systems (measurable, topological, and measurable and topological) which divide sections of the paper. The paper is arranged as follows.

In Section 1 we consider measure preserving systems (MPS) without any topology. The main objective of this section is to characterize almost periodic functions. To do this we introduce a purely measure theoretic notion (no topology) of mean sensitivity with respect to an L^2 function f ($\mu - f-$mean sensitivity see Definition 1.12). We then characterize this concept as the negation of almost periodicity of f (when the action is Abelian). As a corollary, we characterize systems with discrete spectrum as those ergodic systems that are not mean sensitive with respect to all L^2 functions; and we characterize weakly mixing systems as those ergodic systems which are mean sensitive with respect to all non-constant L^2 functions. Note that while it is possible to develop a purely measure theoric version of mean sensitivity, we did not find a natural purely measure theoretic notion of mean equicontinuity (other than negation of mean sensitivity).

In Section 2 we consider topological dynamical systems (TDS) (without measures). We define the purely topological versions of mean equicontinuity and mean sensitivity of a TDS relative to a given continuous function f. We show that in the case of a minimal TDS, mean sensitivity and mean equicontinuity (relative to f) are complementary notions and partition $C(X)$. Finally, for a TDS we give results on the relationships between topological almost periodicity defined by Ellis in [10], and mean equicontinuity. In particular we show that if a function is weakly almost periodic then the system is mean equicontinuous with respect to the function (the converse does not hold).

In Section 3 we consider topological dynamical systems equipped with invariant probability measures. It is in this category where we can define a hybrid (measure theoretic and topological) form of mean equicontinuity relative to a given function f and a measure μ ($\mu - f-$mean equicontinuity). First we show that this notion is complementary to the measure-theoretic notion of mean sensitivity relative to f (defined in Section 1). Finally we present a characterization of discrete spectrum for TDS equipped with ergodic measures (Theorem 3.10). In Section 4 we apply this result to characterize quasicrystalline behaviour on Delone sets. Finally we compare our results with other characterizations of discrete spectrum in terms of topological averages (e.g. [17][21]).

In this paper $\mathbb{G}$ denotes an abelian locally compact group (with operation $+$ and identity e). Every locally compact group has a unique measure that is invariant under rotations, known as Haar measure on $\mathbb{G}$. We will denote this measure with ν. In particular if $\mathbb{G}=\mathbb{R}^d$, then ν is the Lebesgue measure, and if $\mathbb{G}$ countable ν is the cardinality.

A sequence of measurable sets $\{F_n\}_{n\in\mathbb{N}}\subset\mathbb{G}$ with finite non-zero $\nu-$measure is a **Følner sequence** if for every $g\in\mathbb{G}$

$$\lim_{n\to\infty}\frac{\nu(gF_n\triangle F_n)}{\nu(F_n)}=0.$$

We say a group $\mathbb{G}$ is **amenable** if it contains a Følner sequence. Every Abelian group is amenable. In particular if $\mathbb{G}=\mathbb{Z}^d$ or $\mathbb{G}=\mathbb{R}^d$ then $F_n:=[-n,n]^d$ is a Følner sequence.

Definition 0.1. Let $S\subset\mathbb{G}$ and $\{F_n\}_{n\in\mathbb{N}}$ a Følner sequence. We define the **lower density of S** by

$$\underline{D}(S):=\liminf_{n\to\infty}\frac{\nu(S\cap F_n)}{\nu(F_n)},$$

and the **upper density of S** by

$$\overline{D}(S):=\limsup_{n\to\infty}\frac{\nu(S\cap F_n)}{\nu(F_n)}.$$

1. Measure theoretical results.

A $\mathbb{G}-$ **measure preserving system ($\mathbb{G}-$MPS)** is a quadruple (X,Σ,μ,T) where (X,Σ,μ) is a standard probability space and

$$T:\mathbb{G}\times X\to X,(j,x)\to T^jx$$

is a group action (that is $T^ex=x$ and $T^gT^jx=T^{g+j}x\ \forall x\in X$) and $T^j:X\to X$ is measure preserving on (X,Σ,μ) for every $j\in\mathbb{G}$. When it is not needed we will omit writing Σ. A $\mathbb{Z}$-measure preserving system is generated by a measure preserving invertible transformation $T:(X,\mu)\to(X,\mu)$ on a Lebesgue probability space (X,μ).

We say a measurable subset is invariant if it is invariant under every T^i; we say a $\mathbb{G}-$MPS is an **ergodic** if every measurable invariant set has measure 0 or 1. The collection of measurable sets with positive μ-measure is denoted Σ^+.

A measure preserving system (X,μ,T) generates a family of unitary linear operators (known as the Koopman operators) on the complex Hilbert space $L^2(X,\mu)$, $U_T^j:f\mapsto f\circ T^j$. We denote the inner product of $L^2(X,\mu)$ by $\langle\cdot,\cdot\rangle$.

Definition 1.1. Let (X,μ,T) be an ergodic $\mathbb{G}-$MPS and $\{F_n\}$ a Følner sequence.

Let $A\subset X$ be a measurable set. We say $x\in X$ is a **generic point for** A if

$$\lim_{n\to\infty}\frac{\nu(\{i\in F_n:T^ix\in A\})}{\nu(F_n)}=\mu(A).$$

Let f be a measurable function. We say $x \in X$ is a **generic point for** f if

$$\lim_{n\to\infty} \frac{1}{\nu(F_n)} \int_{i\in F_n} f(T^i x) d\nu = \int_X f d\mu(x).$$

Lindenstrauss proved that for every amenable group there exists a Følner sequence $\{F_n\}$ that satisfies the pointwise ergodic theorem i.e. that for every $f \in L^1(X,\mu)$ almost every $x \in X$ is a generic point for f [23].

Remark 1.2. From now on we associate to every $\mathbb{G}$ a Følner sequence $\{F_n\}$ that satisfies the pointwise ergodic theorem.

1.1. **Almost periodic and $\mu-$mean sensitive functions.** Let $S^1 \subset \mathbb{C}$ be the unit circle. Given $\mathbb{G}$ we denote with $\widehat{\mathbb{G}}$ the (Pontryagin) dual group, which is composed by the continuous group homomorphisms from $\mathbb{G}$ to S^1. Given $g \in \mathbb{G}$ and $w \in \widehat{\mathbb{G}}$, we define $(w,j) := w(j)$.

Definition 1.3. Let (X,μ,T) be an ergodic $\mathbb{G}-$MPS and $f \in L^2(X,\mu)$ (complex-valued functions).

$\cdot$We say f is an **almost periodic function** if $cl\left\{U^j f : j \in \mathbb{G}\right\}$ is compact as a subset of $L^2(X,\mu)$. We denote the set of almost periodic functions by H_{ap}.

$\cdot$We say $f \neq 0$ is an **eigenfunction** of (X,μ,T) if there exists $w \in \widehat{\mathbb{G}}$ such that

$$U^j(f) = (w,j)f \quad \forall j \in \mathbb{G}.$$

Remark 1.4. When $d = 1$, the action reduces to the action of a single measure preserving transformation T on (X,μ) and the definition of eigenfunction reduces to the usual definition of an eigenfunction for a unitary operator: $f \neq 0$ and there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and

$$f \circ T = \lambda f.$$

Definition 1.5. A subset $S \subset \mathbb{G}$ is **syndetic** if there exists a bounded set $K \subset \mathbb{G}$ such that $S + K = \mathbb{G}$.

The following results are well known (see for example [12][13]).

Proposition 1.6. *Let (X,μ,T) be a $\mathbb{G}-$MPS.*

1. *H_{ap} is the closure of the set spanned by the eigenfunctions of T.*
2. *$f \in L^2(X,\mu)$ is almost periodic if and only if for every $\varepsilon > 0$ there exists a syndetic set $S \subset \mathbb{G}$ such that $\int \left|f - U^j f\right|^2 d\mu \leq \varepsilon$ for every $j \in S$.*
3. *The product of two almost periodic functions is almost periodic.*
4. *If (X,μ,T) is ergodic and f is an eigenfunction then $|f|$ is constant almost everywhere.*

Definition 1.7. Let (X,μ,T) and (X',μ',T') be two $\mathbb{G}-$MPS. We say they are **isomorphic** if there exists an a.e. bijective and measure preserving function $f : (X,\mu) \to (X',\mu')$ such that $T' \circ f = f \circ T$ and f^{-1} is measure preserving.

Definition 1.8. Let (X,μ,T) be an ergodic $\mathbb{G}-$MPS.

$\cdot(X,\mu,T)$ has **discrete spectrum** if there exists an orthonormal basis for $L^2(X,\mu)$ which consists of eigenfunctions of (X,μ,T).

$\cdot(X,\mu,T)$ has **continuous spectrum** if the only eigenfunctions are the constant functions.

$\cdot(X,\mu,T)$ is a **measurable isometry** if it is isomorphic to (X',μ',T') where (X',T') is where X' is a compact metric space and each element of T' is an isometry.

$\cdot(X,\mu,T)$ is **weakly mixing** if and only if of $(X\times X,\mu\times\mu,T\times T)$ is an ergodic $\mathbb{G}-$MPS.

The following two results are due to Halmos and Von-Neumann (see for example [27]: note that the original proof is stated for $\mathbb{G}=\mathbb{Z}$, nonetheless the same proof is valid for abelian group actions).

Theorem 1.9. *Let (X,μ,T) be an ergodic system. The following are equivalent:*
$\cdot(X,\mu,T)$ *has discrete spectrum.*
$\cdot(X,\mu,T)$ *is a measurable isometry.*
$\cdot L^2(X,\mu)=H_{ap}$

Theorem 1.10. *Let (X,μ,T) be an ergodic system. The following are equivalent:*
$\cdot(X,\mu,T)$ *has continuous spectrum.*
$\cdot(X,\mu,T)$ *is weakly mixing.*
$\cdot H_{ap}$ *consists only of constant functions.*

Let $\mathcal{F}$ denote the set of all functions $\alpha:\mathbb{G}\to\mathbb{C}$ such that

$$\limsup\left(\frac{1}{\nu(F_n)}\int_{F_n}|\alpha(j)|^2\,d\nu(j)\right)<\infty.$$

We define the following pseudometric on $\mathcal{F}$:

$$e(\alpha,\beta):=\limsup\left(\frac{1}{\nu(F_n)}\int_{F_n}|\alpha(j)-\beta(j)|^2\,d\nu(j)\right)^{1/2}$$

It is not difficult to show that this is indeed a pseudometric (i.e. satisfies symmetry and the triangle inequality).

For any $f\in L^2(X,\mu)$ we introduce the following pseudometric on a set of full measure in X.

Definition 1.11. Let $f\in L^2(X,\mu)$. We define

$$d_f(x,y):=\limsup\left(\frac{1}{\nu(F_n)}\int_{F_n}\left|f(T^jx)-f(T^jy)\right|^2d\nu(j)\right)^{1/2}.$$

Using Cauchy-Schwartz and the pointwise ergodic theorem on can check that $d_f(x,y)$ is finite-valued for all $x,y\in X$ that are generic points for f^2. Letting $\alpha(j)=f(T^jx),\beta(j)=f(T^jy)$, we see that $d_f(x,y)=e(\alpha,\beta)$ and thus it is a pseudo-metric on the set of generic points for f^2. Whenever we use the pseudometric d_f we only consider the points that are generic for f^2.

Definition 1.12. Let $f\in L^2(X,\mu)$. We say an $\mathbb{G}-$MPS (X,μ,T) is $\mu-f-$**mean sensitive** if there exists $\varepsilon>0$ such that for every $A\in\Sigma^+$ there exist $x,y\in A$ such that

$$d_f(x,y)>\varepsilon. \tag{1}$$

In this case we also say that f is $\mu-$**mean sensitive** and ε is a μ-**sensitivity constant** for f. We denote the set of $\mu-$mean sensitive functions by H_{ms}.

Remark 1.13. We will see in Section 1.3 that for $f\in L^\infty(X,\mu)$, there are equivalent pseudometrics that give different perspectives on mean sensitivity.

Lemma 1.14. *Let (X,μ,T) be an ergodic $\mathbb{G}-$MPS. Then $H_{ms}\subset L^2(X,\mu)$ is an open set.*

Proof. Let $f \in H_{ms}$ and let $\varepsilon > 0$ be a mean sensitivity constant for f.

Let $g \in L^2(X,\mu)$ such that $\int |f-g|^2 \, d\mu \leq (\varepsilon/4)$. By the pointwise ergodic theorem there exists $Y \subset X$ with $\mu(Y) = 1$ and

$$\lim \frac{1}{|F_n|} \int_{F_n} \left| f(T^j x) - g(T^j x) \right|^2 d\nu(j) \leq (\varepsilon/8) \text{ for every } x \in Y.$$

For every $A \in \Sigma^+$ there exist $x, y \in A \cap Y$ such that

$$\limsup \frac{1}{|F_n|} \int_{F_n} \left| f(T^j x) - f(T^j y) \right|^2 d\nu(j) > \varepsilon.$$

Since $e(\alpha, \beta)$ is a pseudometric we obtain

$$\begin{aligned}
&\limsup \left(\frac{1}{|F_n|} \int_{F_n} \left| g(T^j x) - g(T^j y) \right|^2 d\nu(j) \right)^{1/2} \\
&\geq \limsup \left(\frac{1}{|F_n|} \int_{F_n} \left| f(T^j x) - f(T^j y) \right|^2 d\nu(j) \right)^{1/2} \\
&- \lim \left(\frac{1}{|F_n|} \int_{F_n} \left| f(T^j x) - g(T^j x) \right|^2 d\nu(j) \right)^{1/2} \\
&- \lim \left(\frac{1}{|F_n|} \int_{F_n} \left| f(T^j y) - g(T^j y) \right|^2 d\nu(j) \right)^{1/2} \\
&> \sqrt{\varepsilon}(1 - 1/\sqrt{2}).
\end{aligned}$$

Thus $g \in H_{ms}$ and so $H_{ms} \subset L^2(X,\mu)$ is open. $\square$

Theorem 1.15. *Let (X, μ, T) be an ergodic $\mathbb{G}-$MPS. Then $H_{ap}^c = H_{ms}$.*

Proof. We first show that $H_{ap}^c \subseteq H_{ms}$.

Assume that $f \in H_{ap}^c$, and so $cl \left\{ U^j f : j \in \mathbb{G} \right\}$ is not compact. This implies it is not totally bounded, hence there exists $\varepsilon > 0$ and an infinite subset $S \subset \mathbb{G}$ such that

$$\int \left| U^i f - U^j f \right|^2 d\mu \geq \varepsilon \text{ for every } i \neq j \in S \tag{2}$$

Let $A \in \Sigma^+$. There exist $s \neq t \in S$ such that

$$\mu(T^{-s} A \cap T^{-t} A) > 0.$$

Let $g(x) := \left| U^s f(x) - U^t f(x) \right|^2$. By the pointwise ergodic theorem, there exists $z \in T^{-s} A \cap T^{-t} A$ such that

$$\lim \frac{1}{|F_n|} \int_{F_n} g(T^j z) d\nu(j) = \int g d\mu \geq \varepsilon.$$

Let $p := T^s z \in A$ and $q := T^t z \in A$. Since $\mathbb{G}$ is commutative we have that

$$\begin{aligned}
&\lim \frac{1}{|F_n|} \int_{F_n} \left| f(T^j p) - f(T^j q) \right|^2 d\nu(j) \\
&= \lim \frac{1}{|F_n|} \int_{F_n} \left| f(T^{j+s} z) - f(T^{j+t} z) \right|^2 d\nu(j) \\
&= \lim \frac{1}{|F_n|} \int_{F_n} g(T^j z) d\nu(j) > \varepsilon.
\end{aligned}$$

We conclude that for every $A \in \Sigma^+$ there exist $p, q \in A$ such that

$$\lim \frac{1}{|F_n|} \int_{F_n} \left|f(T^j p) - f(T^j q)\right|^2 d\nu(j) > \varepsilon.$$

Now we show that $H_{ap} \subseteq H_{ms}^c$.

Let $\varepsilon > 0$ and g be a measurable function. There exists a set $X_\varepsilon \in \Sigma_+$ such that $|g(x) - g(y)|^2 \leq \varepsilon$ for every $x, y \in X_\varepsilon$.

Now assume that g is an eigenfunction and so there exists $w \in \mathbb{R}^d$ such that $U^j(g) = e^{2\pi i \langle w,j\rangle} g$. Then for every $x, y \in X_\varepsilon$ and every $j \in \mathbb{G}$ we have that

$$\begin{aligned}
&\left|g(T^j x) - g(T^j y)\right|^2 \\
&= \left|e^{2\pi i\langle w,j\rangle}(g(x) - g(y))\right|^2 \\
&= |g(x) - g(y)|^2 \leq \varepsilon.
\end{aligned}$$

Thus $g \in H_{ms}^c$.

Using Lemma 1.14 and Proposition 1.6 (part 1), we get that $H_{ap} \subset H_{ms}^c$, as desired. □

Note that the proof above shows that in the definition of H_{ms} we can replace lim sup with lim, i.e., $f \in H_{ms}$ if and only if there exists $\varepsilon > 0$ such that for every $A \in \Sigma^+$ there exists $x, y \in A$ such that

$$\lim \frac{1}{|F_n|} \int_{F_n} \left|f(T^j x) - f(T^j y)\right|^2 d\nu(j) > \varepsilon.$$

Corollary 1.16. *Let (X, μ, T) be an ergodic ergodic $\mathbb{G}-$MPS.*

1) *(X, μ, T) has discrete spectrum if and only if $L^2(X, \mu) = H_{ms}^c$*

2) *(X, μ, T) is weakly mixing if and only if $f \in H_{ms}$ for all non-constant $f \in L^2(X, \mu)$.*

Proof. 1) Follows from Theorems 1.9 and 1.15.

2) Follows from Theorems 1.10 and 1.15. □

1.2. **$\mu - f-$Mean expansivity.** The proof of Theorem 1.20 is analogous to the proof of Theorem 2.6 of [14] but uses d_f instead of the Besicovitch pseudometric. The proof is short so we write it for completeness.

Definition 1.17. Let $f \in L^2(X, \mu)$. A $\mathbb{G}-$MPS is $\mu - f-$**mean expansive** if there exists $\varepsilon > 0$ such that $\mu \times \mu(\{(x, y) : d_f(x, y) > \varepsilon\}) = 1$. We say that ε is a μ-**mean expansive constant** for f.

Lemma 1.18. *Let (X, μ, T) be an ergodic $\mathbb{G}-$MPS. Then*

$$g(x) := \mu(\{y : d_f(x, y) \leq \varepsilon\})$$

is constant for almost every $x \in X$ and equal to $\mu \times \mu \{(x, y) : d_f(x, y) \leq \varepsilon\}$.

Proof. It is not hard to see that $d_f(x, y)$ is $\mu \times \mu-$measurable. This means that $\{(x, y) : d_f(x, y) \leq \varepsilon\}$ is $\mu \times \mu-$measurable for every $\varepsilon > 0$. Using Fubini's Theorem we obtain that

$$\begin{aligned}
\mu \times \mu \{(x, y) : d_f(x, y) \leq \varepsilon\} &= \int_X \int_X 1_{\{(x,y):d_f(x,y)\leq\varepsilon\}} d\mu(y) d\mu(x) \\
&= \int_X \mu \{y : d_f(x, y) \leq \varepsilon\} d\mu(x).
\end{aligned}$$

Since g is T-invariant we conclude that $g(x)$ is constant for almost every $x \in X$ and equal to $\mu \times \mu \{(x, y) : d_f(x, y) \leq \varepsilon\}$. □

Definition 1.19. Let $B_\varepsilon^f(x) := \{y : d_f(x, y) \leq \varepsilon\}$.

Theorem 1.20. *Let (X, μ, T) be an ergodic system and $f \in L^2(X, \mu)$. The following are equivalent:*

1) *(X, T) is $\mu - f-$mean sensitive.*
2) *(X, T) is $\mu - f-$mean expansive.*
3) *There exists $\varepsilon > 0$ such that for almost every x, $\mu(B_\varepsilon^f(x)) = 0$.*

Proof. 2) $\Rightarrow$ 1) Let ε be a μ-mean expansive constant for f. Let $A \in \Sigma^+$ and so $A \times A \in (\Sigma \times \Sigma)^+$. By hypothesis we can find $(x, y) \in A \times A$ such that $d_f(x, y) > \varepsilon$.

1) $\Rightarrow$ 3)

Suppose (X, T) is $\mu - f-$mean sensitive (with $\mu - f-$mean sensitivity constant ε) and that 3) is not satisfied. This means there exists $x \in X$ such that $\mu(B_{\varepsilon/2}^f(x)) > 0$. For any $y, z \in B_{\varepsilon/2}^f(x)$, we have that $d_f(y, z) \leq \varepsilon$. This contradicts the assumption that (X, T) is $\mu - f-$mean sensitive.

3) $\Rightarrow$ 2)

Using Lemma 1.18 we obtain that $\mu \times \mu \{(x, y) : d_f(x, y) \leq \varepsilon\} = 0$. □

1.3. **Equivalent pseudometrics.** We consider the following pseudometrics, which turn out to be equivalent to d_f when f is bounded.

Definition 1.21. Let $f \in L^2(X, \mu)$. Define

$$d'_f(x, y) := \limsup \frac{1}{\nu(F_n)} \int_{F_n} \left|f(T^j x) - f(T^j y)\right| d\nu(j)$$

Definition 1.22. Let $f \in L^2(X, \mu)$. Define

$$\rho_f(x, y) := \inf \left\{\varepsilon > 0 : \overline{D}(\Delta_\varepsilon(x, y)) < \varepsilon\right\},$$

where

$$\Delta_\varepsilon(x, y) := \left\{i \in \mathbb{G} : \ |f(T^i x) - f(T^i y)| > \varepsilon\right\}.$$

Proposition 1.23. *For an $\mathbb{G}-$MPS (X, μ, T) and $f \in L^\infty(X, \mu)$, the pseudometrics $d_f(x, y), d'_f(x, y)$, and $\rho_f(x, y)$ are equivalent, i.e., generate the same topology.*

Proof. Without loss of generality we may assume that $|f| \leq 1/2$. So, $(d_f(x, y))^2 \leq d'_f(x, y)$.

Let $\varepsilon > 0$. It suffices to show that 1) if $\rho_f(x, y) < \varepsilon/2$, then $d'_f(x, y) < \varepsilon$ and 2) if $d_f^2(x, y) < \varepsilon^3$, then $\rho_f(x, y) < \varepsilon$.

Proof of 1). If $\rho_f(x, y) < \varepsilon/2$, then

$$\overline{D}(\left\{j \in \mathbb{G} : |f(T^j x) - f(T^j y)| > \varepsilon/2\right\}) < \varepsilon/2.$$

So,

$$\begin{aligned}
&d'_f(x, y) \\
&= \limsup \frac{1}{|F_n|} \int_{F_n} |f(T^j x) - f(T^j y)| d\nu(j) \\
&\leq \frac{\varepsilon}{2} \overline{D}(\left\{j \in \mathbb{G} : |f(T^j x) - f(T^j y)| > \varepsilon/2\right\}) + \overline{D}(\left\{j \in \mathbb{G} : |f(T^j x) - f(T^j y)| \leq \varepsilon/2\right\}) \\
&< \varepsilon.
\end{aligned}$$

Proof of 2). Assume that $d_f^2(x,y) < \varepsilon^3$. Suppose that $\rho_f(x,y) \geq \varepsilon$. Then for all $\delta < \varepsilon$, we obtain

$$\begin{aligned}
&\limsup \frac{1}{|F_n|} \int_{F_n} |f(T^j x) - f(T^j y)|^2 d\nu(j) \\
&\geq \delta^2 \overline{D}(\{j \in \mathbb{G} : |f(T^j x) - f(T^j y)|^2 > \delta^2\}) \\
&= \delta^2 \overline{D}(\{j \in \mathbb{G} : |f(T^j x) - f(T^j y)| > \delta\}) \\
&\geq \delta^3.
\end{aligned}$$

Thus

$$\limsup \frac{1}{|F_n|} \int_{F_n} |f(T^j x) - f(T^j y)|^2 d\nu(j) \geq \varepsilon^3$$

yields a contradiction. We conclude that $\rho_f(x,y) < \varepsilon$. □

This implies we can use $d_f'(x,y)$ or $\rho_f(x,y)$ as alternatives to $d_f(x,y)$ in the definition of μ-f-mean sensitivity.

2. **Topological results.**

A $\mathbb{G}-$**topological dynamical system (TDS)** is a pair (X,T), where X is a compact metric space and T is a group action with $T^j : X \to X$ continuous for every $j \in \mathbb{G}$.

The closed ε-balls of X will be denoted by $B_\varepsilon(x)$, and the collection of Borel sets of X by $\mathcal{B}_X$.

2.1. **Mean equicontinuity and mean sensitivity.**

Definition 2.1. We define

$$d_b(x,y) := \limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n} d(T^j x, T^j y) d\nu(j),$$

$$\rho_b(x,y) := \inf\left\{\varepsilon > 0 : \overline{D}(\Delta_\varepsilon(x,y)) < \varepsilon\right\},$$

where

$$\Delta_\varepsilon(x,y) := \left\{i \in \mathbb{G} :\ d(T^i x, T^i y) > \varepsilon\right\}.$$

Using subadditivity of lim sup, it is not hard to show that d_b and ρ_b are indeed pseudometrics. The subscript "b" stands for "Besicovitch," as these are versions of the Besicovitch pseudometric.

Lemma 2.2. *Let (X,T) be a $\mathbb{G}-$TDS. Then d_b and ρ_b are equivalent pseudometrics.*

Proof. The proof is nearly identical to the proof of equivalence of d_f' and ρ_f in Proposition 1.23: simply replace $|f(T^i x) - f(T^i y)|$ by $d(T^i x, T^i y)$ in the proof. □

Definition 2.3. Let (X,T) be a $\mathbb{G}-$TDS. We say (X,T) is **mean sensitive** if for every non-empty open set U there exist $x,y \in U$ such that $d_b(x,y) > \varepsilon$.

Definition 2.4. Let (X,T) be a $\mathbb{G}-$TDS. We say $x \in X$ is a **mean equicontinuity point** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in B_\delta(x)$ then

$$d_b(x,y) < \varepsilon.$$

We say (X,T) is **mean equicontinuous (or mean-L-stable)** if every $x \in X$ is a mean equicontinuity point. We say (X,T) is **almost mean equicontinuous** if the set of mean equicontinuity points is residual.

Using Lemma 2.2, we conclude that (X,T) is mean sensitive if and only if for every non-empty open set U there exist $x, y \in U$ such that $\rho_b(x,y) > \varepsilon$; and $x \in X$ is a mean equicontinuity point if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in B_\delta(x)$ then

$$\rho_b(x,y) < \varepsilon.$$

Remark 2.5. If (X,T) is mean equicontinuous, then it is uniformly mean equicontinuous in the sense that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then $\overline{D}(i \in \mathbb{G} : d(T^i x, T^i y) > \varepsilon) < \varepsilon$. One way to see this is by identifying points x, y such that $d_b(x,y) = 0$. Namely, we obtain a metric space $(X/d_b, d_b)$ and a natural projection $\pi_b : (X,d) \to (X/d_b, d_b)$ by mapping $x \in X$ to its equivalence class in X/d_b. Then mean equicontinuity of (X,T) means that π_b is continuous and hence uniformly continuous.

Mean equicontinuous systems were introduced as mean-L-stable systems by Fomin [11]. These systems have been studied in [2], [24], [26] and [22].

In [1][3] dichotomies between equicontinuity and sensitivity were exhibited. In [14] (and independently in [22]) the following dichotomies involving mean sensitivity and mean equicontinuity were proved.

Definition 2.6. Let (X,T) be a $\mathbb{G}-$TDS. We say $x \in X$ is a **transitive point** if $\left\{T^i x : i \in \mathbb{G}\right\}$ is dense. We say (X,T) is **transitive** if X contains a residual set of transitive points. If every $x \in X$ is transitive then we say the system is **minimal**.

Theorem 2.7. [14][22]*A transitive system is either almost mean equicontinuous or mean sensitive. A minimal system is either mean equicontinuous or mean sensitive.*

In the next subsection we will introduce versions of mean equicontinuity and mean sensitivity relative to given $f \in C(X)$ and we will establish similar dichotomies.

2.2. $f-$mean equicontinuity and $f-$mean sensitivity.

Definition 2.8. Let (X,T) be a $\mathbb{G}-$TDS and $f \in C(X)$. We say (X,T) is $f-$**mean sensitive** if there exists $\varepsilon > 0$ such that for every non-empty open set U there exists $x, y \in U$ such that

$$d_f(x,y) > \varepsilon.$$

In this case we also say that f is **mean sensitive** for (X,T), and we denote the set of all mean sensitive functions for (X,T) by C_{ms}.

The notion of $f-$mean equicontinuity appeared in [26].

Definition 2.9. Let (X,T) be a $\mathbb{G}-$TDS and $f \in C(X)$. We say $x \in X$ is a $f-$**mean equicontinuity point** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then $d_f(x,y) \leq \varepsilon$. We say (X,T) is $f-$**mean equicontinuous** if all $x \in X$ are $f-$mean equicontinuous. In this case we also say that f is **mean equicontinuous function** for (X,T); we denote the set of mean equicontinuous functions by C_{me}. We say (X,T) is $f-$ **almost mean equicontinuous** if the set of mean equicontinuity points is residual.

Remark 2.10. Since any continuous function on X is bounded, we can use Proposition 1.23, to characterize $f-$mean equicontinuity and $f-$mean sensitivity by replacing $d_f(x,y)$ with $\rho_f(x,y)$ in the definition. We will mainly use the latter.

Again using the idea that a continuous function on a compact metric set is uniformly continuous, it is not hard to see that if (X,T) is $f-$mean equicontinuous, then it is uniformly $f-$mean equicontinuous in the sense that for every $\varepsilon>0$ there exists $\delta>0$ such that if $d(x,y)\leq\delta$ then $d_f(x,y)\leq\varepsilon$.

Definition 2.11. Let $f\in C(X)$. We denote the set of $f-$mean equicontinuity points by E^f and we define

$$E^f_\varepsilon := \left\{x\in X : \exists\delta>0\ \forall y,z\in B_\delta(x),\ \underline{D}\left\{i\in\mathbb{G} : \left|f(T^iy)-f(T^iz)\right|\leq\varepsilon\right\}\geq 1-\varepsilon\right\}.$$

Note that $E^f=\cap_{\varepsilon>0}E^f_\varepsilon$.

Lemma 2.12. *Let (X,T) be a $\mathbb{G}-$TDS. The sets E^f, E^f_ε are inversely invariant (i.e. $T^{-j}(E^f)\subseteq E^f$, $T^{-j}(E^f_\varepsilon)\subseteq E^f_\varepsilon$ for all $j\in\mathbb{G}$) and E^f_ε is open.*

Proof. Let $j\in\mathbb{G}$, $\varepsilon>0$, and $x\in T^{-j}E^f_\varepsilon$. There exists $\eta>0$ such that if $d(T^jx,z)\leq\eta$ and $d(T^jx,y)\leq\eta$ then

$$\underline{D}\left\{i : \left|f(T^iy)-f(T^iz)\right|\leq\varepsilon\right\}\geq 1-\varepsilon.$$

There also exists $\delta>0$ such that if $d(x,y)\leq\delta$ then $d(T^jx,T^jy)\leq\eta$. So, if $y,z\in B_\delta(x)$, then $\underline{D}\left\{i : \left|f(T^{i+j}y)-f(T^{i+j}z)\right|\leq\varepsilon\right\}\geq 1-\varepsilon$. We conclude that $x\in E^f_\varepsilon$, and so E^f_ε is inversely invariant. It follows that E^f is also inversely invariant.

Let $x\in E^f_\varepsilon$ and $\delta>0$ satisfy the defining property of E^f_ε. If $d(x,w)<\delta/2$ then $w\in E^f_\varepsilon$; indeed if $y,z\in B_{\delta/2}(w)$ then $y,z\in B_\delta(x)$. So, E^f_ε is open. $\square$

It is not hard to see that $f-$mean sensitive systems have no $f-$mean equicontinuity points. The proof of the following dichotomies is very similar to the proof of Theorem 2.7; we include it for completeness.

Theorem 2.13. *Let $f\in C(X)$. A transitive system is either $f-$almost mean equicontinuous or $f-$mean sensitive. A minimal system is either $f-$mean equicontinuous or $f-$mean sensitive.*

Proof. First, we show that if (X,T) is a transitive system then for every ε, E^f_ε is either empty or dense. Assume E^f_ε is non-empty and not dense. Then $U=X\diagdown(cl(E^f_\varepsilon))$ is a non-empty open set. Since the system is transitive and E^f_ε is nonempty and open (by Lemma 2.12) there exists $t\in\mathbb{G}$ such that $U\cap T^{-t}(E^f_\varepsilon)$ is non empty. By Lemma 2.12 we have that $U\cap T^{-t}(E^f_\varepsilon)\subset U\cap E^f_\varepsilon=\emptyset$, a contradiction.

If E^f_ε is non-empty for every $\varepsilon>0$ then we have that $E^f=\cap_{n\geq1}E^f_{1/n}$ is a residual set; hence the system is f-almost mean equicontinuous.

If there exists $\varepsilon>0$ such that E^f_ε is empty, then for any open set U there exist $y,z\in U$ such that $\underline{D}\left\{i\in\mathbb{G} : \left|f(T^iy)-f(T^iz)\right|\leq\varepsilon\right\}<1-\varepsilon$; this means that $\overline{D}\left\{i\in\mathbb{G} : \left|f(T^iy)-f(T^iz)\right|>\varepsilon\right\}\geq\varepsilon$. It follows that (X,T) is $f-$mean sensitive.

Now suppose (X,T) is minimal and f-almost mean equicontinuous. For every $x\in X$ and every $\varepsilon>0$ there exists $t\in\mathbb{G}$ such that $T^tx\in E^f_\varepsilon$. Since E^f_ε is inversely invariant, we have $x\in E^f_\varepsilon$. So, $x\in E^f$. $\square$

In the following result we use a technique from the topological Halmos Von-Neumann Theorem (e.g. see Chapter 5.5 [27]).

Theorem 2.14. *Let (X,T) be a $\mathbb{G}-$TDS. Then (X,T) is mean equicontinuous if and only if it is $f-$mean equicontinuous for every $f\in C(X)$ (i.e. $C(X)=C_{me}$).*

Proof. $\Rightarrow$

Let (X,T) be mean equicontinuous, $f \in C(X)$ and $\varepsilon > 0$. Since X is compact, f is uniformly continuous; thus there exists $\delta' \in (0,\varepsilon)$ such that if $d(x,y) \leq \delta'$ then $|f(x) - f(y)| \leq \varepsilon$. Using that (X,T) is mean equicontinuous there exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then

$$\underline{D}(i \in \mathbb{G} : d(T^i x, T^i y) \leq \delta') \geq 1 - \delta'.$$

This implies that

$$\underline{D}(i \in \mathbb{G} : |f(x) - f(y)| \leq \varepsilon) \geq 1 - \varepsilon.$$

Hence (X,T) is $f-$mean equicontinuous.

$\Leftarrow$

Let $\{f_n\}$ be a collection of functions such that $|f_n| \leq 1$ and the closure of its linear span is $C(X)$.

Such a collection separates points of X, and so

$$\overline{d}(x,y) := \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n}$$

is a metric on X. Let $\varepsilon > 0$ and choose N so that $\sum_{n=N+1}^{\infty} 2/2^n \leq \varepsilon/2$. There exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then

$$|f_n(x) - f_n(y)| \leq \varepsilon/2$$

for every $1 \leq n \leq N$. Thus if $d(x,y) \leq \delta$ then

$$\sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So, the identity map from (X,d) to $(X,\overline{d})$ is a bijective continuous map on a compact metric space and hence is a homeomorphism. We conclude that d and $\overline{d}$ are equivalent metrics.

It is not hard to see that mean equicontinuity is invariant under a change of equivalent metrics. We will show (X,T) is mean equicontinuous with respect to $\overline{d}$.

Let $\varepsilon > 0$ and choose N so that $\sum_{n=N+1}^{\infty} 2/2^n \leq \varepsilon/2$. There exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then

$$\limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n} \left| f_m(T^i x) - f_m(T^i y) \right| d\nu(i) \leq \varepsilon/2$$

for every $1 \leq m \leq N$. Since d and $\overline{d}$ are equivalent metrics, there exists $\eta > 0$ such that if $\overline{d}(x,y) \leq \eta$, then $d(x,y) \leq \delta$, and so

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n} \overline{d}(T^i x, T^i y) d\nu(i) \\
&\leq \varepsilon/2 + \limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n} \sum_{m=1}^{N} \frac{\left| f_m(T^i x) - f_m(T^i y) \right|}{2^m} d\nu(i) \\
&\leq \varepsilon.
\end{aligned}$$

So, (X,T) is mean equicontinuous with respect to $\overline{d}$. $\square$

2.3. **Weakly almost periodic functions.** In [10], Ellis defined two topological notions of almost periodic functions in $C(X)$.

We consider $C(X)$ as a Banach space with the ℓ^∞ norm. Two classic topologies studied on Banach spaces are: the strong topology (given by the norm) and the weak topology (the coarsest topology such that each element in the dual space is continuous). Since X is compact $f_n \to f$ in the weak topology if and only if $f_n(x) \to f(x)$ for every $x \in X$.

Definition 2.15. Let (X,T) be a $\mathbb{G}-$TDS. We say $f \in C(X)$ is **almost periodic** if $\left\{U^j f : j \in \mathbb{G}\right\} \subset C(X)$ is precompact with respect to the strong topology. We say $f \in C(X)$ is **weakly almost periodic** (WAP) if $\left\{U^j f : j \in \mathbb{G}\right\} \subset C(X)$ is precompact with respect to the weak topology. If every $f \in C(X)$ is WAP then we say (X,T) is **weakly almost periodic**.

A system is equicontinuous if and only if every function is almost periodic. A minimal system is equicontinuous if and only if it is weakly almost periodic [10]; nonetheless there are examples of transitive weakly almost periodic systems that are not equicontinuous. We will show these systems are always mean equicontinuous.

Lemma 2.16. *Let (X,T) be a $\mathbb{G}-$TDS. If f is almost periodic then it is mean equicontinuous.*

Proof. Let $\varepsilon > 0$. There exists a finite set $F \subset \mathbb{G}$ such that for every $j \in \mathbb{G}$ there exists $i_j \in F$ such that $\left|U^j f - U^{i_j} f\right|_\infty \leq \varepsilon$ (where U is the Koopman operator on $C(X)$). By uniform continuity there exists $\delta > 0$ such that if $d(x,y) \leq \delta$ then $\left|U^i f(x) - U^i f(y)\right| \leq \varepsilon$ for all $i \in F$. Thus, if $d(x,y) \leq \delta$ then for every j

$$\begin{aligned}
&\left|f(T^j x) - f(T^j y)\right| \\
&\leq \left|f(T^j x) - f(T^{i_j} x)\right| + \left|f(T^{i_j} x) - f(T^{i_j} y)\right| + \left|f(T^{i_j} y) - f(T^j y)\right| \\
&\leq 3\varepsilon
\end{aligned}$$

and thus

$$\begin{aligned}
&d'_f(x,y) = \\
&\limsup \frac{1}{|F_n|} \int_{F_n} \left|f(T^j x) - f(T^j y)\right| d\nu(j) \\
&\leq 3\varepsilon.
\end{aligned}$$

By the equivalence of the metrics, $d_f(x,y)$ and $d'_f(x,y)$, we conclude that f is mean equicontinuous. ∎

Proposition 2.17. *Let $\mathbb{G}$ be a countable group and (X,T) be a uniquely ergodic $\mathbb{G}-$TDS. If f is weakly almost periodic then it is mean equicontinuous.*

Proof. Let $f \in C(X)$ be weakly almost periodic. As a consequence of the de Leeuw-Glicksberg decomposition (see Theorem 1.51 in [16]) we have that

$$f = g + h,$$

where g is almost periodic and $\lim \frac{1}{|F_n|} \sum_{i \in F_n} h(T^i x) = 0$ for every x. This implies h is mean equicontinuous. By the previous lemma g is mean equicontinuous; we conclude f is mean equicontinuous. ∎

Using that transitive WAP systems are uniquely ergodic (Lemma 1.50 in [16]), the previous Proposition and Theorem 2.14 we obtain the following result.

Corollary 2.18. *Let $\mathbb{G}$ be a countable group. If (X,T) is a transitive WAP $\mathbb{G}-$TDS then it is mean equicontinuous.*

3. **Measure theoretic and topological results.** In this section we present hybrid results, that is, results that reflect the topology as well as the measure theoretic structure of the sytem.

We remind the reader that given a metric space X and a Borel probability measure μ we denote the Borel sets with $\mathcal{B}_X$ and the Borel sets with positive measure with $\mathcal{B}_X^+$.

We say (X,μ,T) is an **ergodic $\mathbb{G}-$TDS** if (X,T) is a $\mathbb{G}-$TDS, μ a Borel probability measure and (X,μ,T) an ergodic $\mathbb{G}-$MPS.

3.1. **$\mu-$mean equicontinuity.**

Definition 3.1. Let (X,T) be a $\mathbb{G}-$TDS, μ an invariant Borel probability measure. We say (X,T) is **$\mu-$mean equicontinuous** if for every $\tau>0$ there exists a compact set $M\subset X$, with $\mu(M)\geq 1-\tau$, such that for every $\varepsilon>0$ there exists $\delta>0$ such that whenever $x,y\in M$ and $d(x,y)\leq\delta$ then

$$d_b(x,y)\leq\varepsilon.$$

Remark 3.2. It is not hard to see that $d_b(\cdot,\cdot)$ is a Borel function.

Also, denoting the closed $\varepsilon-$balls of the pseudometric d_b by $B_\varepsilon^b(x)$, for every $\varepsilon>0$ and every $x\in X$ we have that $B_\varepsilon^b(x)$ is $\mu-$measurable.

Measure theoretic forms of sensitivity for $\mathbb{G}-$TDS have been studied in [15],[7], and [18]. In particular in [18] it was shown that ergodic $\mathbb{G}-$TDS are either $\mu-$equicontinuous or $\mu-$sensitive. In [14] $\mu-$mean sensitivity was introduced.

Definition 3.3. A $\mathbb{G}-$TDS (X,T) is **$\mu-$mean sensitive** if there exists $\varepsilon>0$ such that for every $A\in\mathcal{B}_X^+$ there exists $x,y\in A$ such that

$$d_b(x,y)>\varepsilon.$$

A $\mathbb{G}-$TDS (X,T) is **$\mu-$mean expansive** if there exists $\varepsilon>0$ such that $\mu\times\mu\left\{(x,y):d_b(x,y)>\varepsilon\right\}=1$.

Theorem 3.4. [14]*Let (X,μ,T) be an ergodic $\mathbb{G}-$TDS. The following are equivalent:*

1) *(X,T) is $\mu-$mean sensitive.*
2) *(X,T) is $\mu-$mean expansive.*
3) *There exists $\varepsilon>0$ such that for almost every x, $\mu(B_\varepsilon^b(x))=0$.*
4)*(X,T) is not $\mu-$mean equicontinuous.*

In the next subsection we will obtain a similar result for $\mu-f-$mean sensitivity.

3.2. **$\mu-f-$mean equicontinuity.** For the definition of d_f and B_ε^f see Definition 1.12 and Definition 1.19.

Definition 3.5. Let (X,T) be a $\mathbb{G}-$TDS, μ a Borel probability measure and $f\in L^2(X,\mu)$. We say (X,T) is **$\mu-f-$mean equicontinuous** if for every $\tau>0$ there exists a compact set $M\subset X$, with $\mu(M)\geq 1-\tau$, such that for every $\varepsilon>0$ there exists $\delta>0$ such that whenever $x,y\in M$ and $d(x,y)\leq\delta$ then

$$\rho_f(x,y)\leq\varepsilon.$$

In this case we say f is **$\mu-$mean equicontinuous**. We denote the set of $\mu-$mean equicontinuous functions by H_{me}.

The following fact is well known (for a proof see [14]).

Lemma 3.6. *Let (Y, d_Y) be a metric space. Suppose that there is no uncountable set $A \subset Y$ and $\varepsilon > 0$ such that $d_Y(x, y) > \varepsilon$ for every $x, y \in A$ with $x \neq y$, then (Y, d_Y) is separable.*

Theorem 3.7. *Let (X, μ, T) be an ergodic $\mathbb{G}-$TDS and $f \in L^2(X, \mu)$. The following are equivalent:*
1) *(X, T) is $\mu - f-$mean sensitive.*
2) *(X, T) is $\mu - f-$mean expansive.*
3) *There exists $\varepsilon > 0$ such that for almost every x, $\mu(B^f_\varepsilon(x)) = 0$.*
4)*(X, T) is not $\mu - f-$mean equicontinuous.*

Proof. 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3)
Given by Theorem 1.20.
The following proofs are similar to the proof of Theorem 3.4.
2) $\Rightarrow$ 4)
If (X, T) is μ-f-mean-expansive, then there is a set $Z \subset X \times X$ s.t $\mu(Z) = 1$ and for all $(x, y) \in Z$, we have $d_f(x, y) > \varepsilon$. Suppose that (X, T) is also μ-f-mean equicontinuous. Then there is a compact set $M \subseteq X$ with positive measure and $\delta > 0$ such that if $x, y \in M$ and $d(x, y) \leq \delta$, then $d_f(x, y) \leq \varepsilon$. Since M is compact, it can be covered by finitely many $\delta/2$-balls. For at least one of these balls B, we have $\mu(B \cap M) > 0$ and so $((B \cap M) \times (B \cap M)) \cap Z \neq \emptyset$. For any (x, y) in this intersection, we have $d_f(x, y) \leq \varepsilon$ and $d_f(x, y) > \varepsilon$, a contradiction.
4) $\Rightarrow$ 3)
Suppose 3) is not satisfied. By Lemma 1.18 we have that for every $n \in \mathbb{N}$ there exists a set of full measure Y_n such that $\mu(B^f_{1/n}(x)) > 0$ for all $x \in Y_n$. Let $Y := \cap_{n\in\mathbb{N}} Y_n$. If $(Y \diagup d_f, d_f)$ is not separable then by Lemma 3.6 there exists $\varepsilon > 0$ and an uncountable set A such that for every $x, y \in A$ with $x \neq y$ we have that $B^f_\varepsilon(x) \cap B^f_\varepsilon(y) = \emptyset$. For any $1/n < \varepsilon$, this is a contradiction to the fact that $\mu(B^f_\varepsilon(x)) > 0$ for all $x \in Y_n$. Hence, $(Y \diagup d_f, d_f)$ is separable. Using Lusin's Theorem we conclude (X, T) is $\mu - f-$mean equicontinuous (this is done exactly as in Proposition 19 in [14]). □

Corollary 3.8. *Let (X, μ, T) be an ergodic $\mathbb{G}-$TDS. Then $H_{ap} = H^c_{ms} = H_{me}$.*

Definition 3.9. We say a function $f \in L^2(X, \mu)$ is simple if it is a linear combination of indicator functions (of measurable sets) 1_B.

In the following result, the equivalence of 1), 4) and 5) was proved in [14] for $\mathbb{Z}^d$ systems. That proof used symbolic systems and does not work for continuous groups like $\mathbb{R}^d$.

Theorem 3.10. *Let (X, μ, T) be an ergodic $\mathbb{G}-$TDS. The following conditions are equivalent:*
1) *(X, T) is $\mu-$mean equicontinuous*
2) *(X, T) is $\mu - 1_B-$mean equicontinuous for every $B \in \mathcal{B}_X$.*
3) *(X, T) is $\mu - f-$mean equicontinuous for every $f \in L^2$.*
4) *(X, μ, T) has discrete spectrum.*
5) *(X, T) is not $\mu-$mean sensitive.*

Proof. 2) $\Leftrightarrow$ 3)
By Corollary 3.8, $H_{me} = H_{ap}$ and thus by Proposition 1.6 it is a closed subspace of $L^2(X, \mu)$. Now use the fact that simple functions are dense in L^2.

3) $\Leftrightarrow$ 4)

Follows from Theorem 1.9 and Corollary 3.8.

1) $\Leftrightarrow$ 5)

This is part of Theorem 3.4.

4) $\Rightarrow$ 1)

If (X, μ, T) has discrete spectrum, then it is measure-theoretically isomorphic to an isometry (by Theorem 1.9). Isometries are equicontinuous and therefore $\mu-$mean equicontinuous. Now use the fact that $\mu-$mean equicontinuity is an isomorphism invariant (see Proposition 28 in [14]).

1) $\Rightarrow$ 2)

Let $B \in \mathcal{B}_X$ and $1 > \varepsilon > 0$.

There exists a compact set M_1 with $\mu(M_1) \geq 1 - \varepsilon/3$ and $\varepsilon' > 0$ such that if $x, y \in M_1$ and $d(x, y) \leq \varepsilon'$ then $1_B(x) = 1_B(y)$ (namely, M_1 is the union of two compact sets, one approximating B and the other approximating B^c, both from the inside). We may assume that $\varepsilon' \leq (1/3)\varepsilon$.

Since (X, T) is $\mu-$mean equicontinuous there exists a compact set M_2 with $\mu(M_2) \geq 1 - \varepsilon/3$ and $\delta > 0$ such that if $x, y \in M_2$ and $d(x, y) \leq \delta$ then

$$\underline{D}(i \in \mathbb{G} : d(T^i x, T^i y) \leq \varepsilon') \geq 1 - \varepsilon'.$$

Let $M = M_1 \cap M_2 \cap \{\text{generic points for } M_1\}$ and $x, y \in M$. We have that $\mu(M) \geq 1 - \varepsilon$. Since $\varepsilon < 1$, we have

$$1_B T^i x = 1_B T^i y \text{ if and only if } |1_B T^i x - 1_B T^i y| \leq \varepsilon.$$

So, if $d(x, y) \leq \delta$ then

$$\begin{aligned}
&\underline{D}(i \in \mathbb{G} : |1_B T^i x - 1_B T^i y| \leq \varepsilon) \\
&= \underline{D}(i \in \mathbb{G} : 1_B T^i x = 1_B T^i y) \\
&\geq \underline{D}(i \in \mathbb{G} : d(T^i x, T^i y) \leq \varepsilon' \text{ and } T^i x, T^i y \in M_1) \\
&\geq 1 - \varepsilon' - 2\varepsilon/3 \\
&\geq 1 - \varepsilon.
\end{aligned}$$

We conclude that (X, T) is $\mu - 1_B-$mean equicontinuous. □

4. **Relationship to results of quasicrystals.** Of particular interest in the mathematical theory of quasicrystals and aperiodic order is studying long range order that Delone sets may exhibit; Delone sets are uniformly discrete and relatively dense subsets of $\mathbb{R}^d$ (see [4] and [19] for recent general expositions that contain the main definitions in this section). A Delone set is crystalline if it is periodic (in all d directions). These are the most ordered Delone sets. We say a Delone set is quasicrystalline if it has pure point diffraction spectrum (this notion is defined using Fourier transforms) and uniform patch frequency (configurations of patches of points appear with a uniform frequency). Periodic Delone sets always have pure point diffraction spectrum and uniform patch frequency. Initially it was not known if non-periodic subsets could have pure point diffraction spectrum; nowadays there are many examples, perhaps the best known is the Delone set associated to the Penrose tiling [25]. To every Delone set we can associate an $\mathbb{R}^d-$topological dynamical system (uniquely ergodic when the set has uniform patch frequency), where the $\mathbb{R}^d-$action is defined by the shifts in the d directions. In [6] Baake, Lenz and Moody showed that a Delone set is crystalline if and only if the associated $\mathbb{R}^d$- topological dynamical system is equicontinuous (this means that the family of the shifts is

equicontinuous). This motivates the following question; can quasicrystals be characterized with a weaker forms of equicontinuity? Combining the work of several papers it is known that a Delone set is quasicrystalline if and only if the associated dynamical system is uniquely ergodic and has discrete spectrum [9, 20, 5]. Using this and Theorem 3.10 we obtain that a Delone set is quasicrystalline if and only if the associated dynamical system is uniquely ergodic and μ-mean equicontinuous (with μ the unique invariant measure).

There are other characterizations that use the Besicovitch pseudometric to characterize discrete spectrum. In [17] it is proven that a (minimal) uniquely ergodic Delone dynamical system has discrete spectrum if and only if it is mean almost periodic. That is, for every $x \in X$ and $\varepsilon > 0$ there exists a syndetic set $S \subset \mathbb{R}^d$, such that $d_b(T^i x, x) \leq \varepsilon$ for every $i \in S$. This result is different from ours in the following senses; besides the clear distinction that mean almost periodicity and mean equicontinuity are different concepts we see that this result applies only for minimal systems and the condition mean almost periodicity does not depend on the measure. Our characterization using $\mu-$mean equicontinuity requires the property of mean equicontinuity only on sets with large measure. Nonetheless there is a relationship. It is not hard to see that every minimal mean equicontinuous $\mathbb{G}-$TDS is mean almost periodic. Let $\varepsilon > 0$ and $\delta > 0$ the number such that if $d(x,y) \leq \delta$ then $d_b(x,y) \leq \varepsilon$. Since (X,T) is minimal every point is almost periodic, thus there exists a syndetic set $S \subset \mathbb{R}^d$, such that $d(T^i x, x) \leq \delta$ for every $i \in S$; this implies $d_b(T^i x, x) \leq \varepsilon$ for every $i \in S$. On [21] there is a similar characterization that uses Bohr almost periodicity and applies to general $\mathbb{G}-$TDS.

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REFERENCES

[1] E. Akin, J. Auslander and K. Berg, When is a transitive map chaotic?, *Ohio State Univ. Math. Res. Inst. Publ.*, **5** (1996), 25–40.
[2] J. Auslander, Mean- l-stable systems, *Illinois Journal of Mathematics*, **3** (1959), 566–579.
[3] J. Auslander and J. A. Yorke, Interval maps, factors of maps, and chaos, *Tohoku Mathematical Journal*, **32** (1980), 177–188.
[4] M. Baake and U. Grimm, *Aperiodic Order*, Cambridge University Press, 2013.
[5] M. Baake and D. Lenz, Dynamical systems on translation bounded measures: Pure point dynamical and diffraction spectra, *Ergodic Theory and Dynamical Systems*, **24** (2004), 1867–1893.
[6] M. Baake, D. Lenz and R. V. Moody, Characterization of model sets by dynamical systems, *Ergodic Theory and Dynamical Systems*, **27** (2007), 341–382.
[7] B. Cadre and P. Jacob, On pairwise sensitivity, *Journal of Mathematical Analysis and Applications*, **309** (2005), 375–382.
[8] T. Downarowicz and E. Glasner, Isomorphic extensions and applications, *Topological Methods in Nonlinear Analysis,* **48** (2016), 321–338.
[9] S. Dworkin, Spectral theory and x-ray diffraction, *Journal of mathematical physics*, **34** (1993), 2965–2967.
[10] R. Ellis, Equicontinuity and almost periodic functions, *Proceedings of the American Mathematical Society*, **10** (1959), 637–643.
[11] S. Fomin, On dynamical systems with pure point spectrum (russian), *Dokl. Akad. Nauk SSSR*, **77** (1951), 29–32.
[12] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, N.J., 1981.
[13] H. Furstenberg, Y. Katznelson and D. Ornstein, The ergodic theoretical proof of szemerédi's theorem, *Bulletin of the AMS*, **7** (1982), 527–552.

[14] F. García-Ramos, Weak forms of topological and measure theoretical equicontinuity: Relationships with discrete spectrum and sequence entropy, *Ergodic Theory and Dynamical Systems*, **37** (2017), 1211–1237.

[15] R. H. Gilman, Classes of linear automata, *Ergodic Theory Dyn. Syst.*, **7** (1987), 105–118.

[16] E. Glasner, *Ergodic Theory Via Joinings*, American Mathematical Society, Providence, RI, 2003.

[17] J.-B. Gouere, Quasicrystals and almost periodicity, *Communications in Mathematical Physics*, **255** (2005), 655–681.

[18] W. Huang, P. Lu and X. Ye, Measure-theoretical sensitivity and equicontinuity, *Israel Journal of Mathematics*, **183** (2011), 233–283.

[19] J. Kellendonk, D. Lenz and J. Savinien, *Mathematics of Aperiodic Order*, Progress in Mathematics, 309. Birkh?user/Springer, Basel, 2015.

[20] J.-Y. Lee, R. V. Moody and B. Solomyak, Pure point dynamical and diffraction spectra, *Annales Henri Poincaré*, **3** (2002), 1003–1018.

[21] D. Lenz, An autocorrelation and discrete spectrum for dynamical systems on metric spaces, arXiv preprint, `arXiv:1608.05636`.

[22] J. Li, S. Tu and X. Ye, Mean equicontinuity and mean sensitivity, *Ergodic Theory and Dynamical Systems*, **35** (2015), 2587–2612.

[23] E. Lindenstrauss, Pointwise theorems for amenable groups, *Inventiones mathematicae*, **146** (2001), 259–295.

[24] J. C. Oxtoby, Ergodic sets, *Bulletin of the American Mathematical Society*, **58** (1952), 116–136.

[25] E. Robinson Jr, The dynamical properties of penrose tilings, *Transactions of the American Mathematical Society*, **348** (1996), 4447–4464.

[26] B. Scarpellini, Stability properties of flows with pure point spectrum, *Journal of the London Mathematical Society*, **26** (1982), 451–464.

[27] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

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