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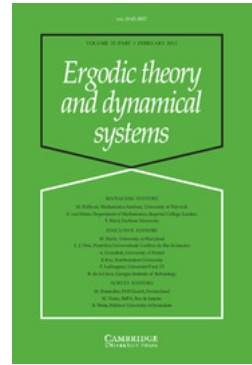
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BRIAN MARCUS and RONNIE PAVLOV

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Approximating entropy for a class of \mathbb{Z}^2 Markov random fields and pressure for a class of functions on \mathbb{Z}^2 shifts of finite type

BRIAN MARCUS and RONNIE PAVLOV

*Department of Mathematics, University of British Columbia, 1984 Mathematics Road,
Vancouver, BC V6T 1Z2, Canada
(e-mail: marcus@math.ubc.ca, rpavlov@du.edu)*

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Abstract. For a class of \mathbb{Z}^2 Markov Random Fields (MRFs) μ , we show that the sequence of successive differences of entropies of induced MRFs on strips of height n converges exponentially fast (in n) to the entropy of μ . These strip entropies can be computed explicitly when μ is a Gibbs state given by a nearest-neighbor interaction on a strongly irreducible nearest-neighbor \mathbb{Z}^2 shift of finite type X . We state this result in terms of approximations to the (topological) pressures of certain functions on such an X , and we show that these pressures are computable if the values taken on by the functions are computable. Finally, we show that our results apply to the hard core model and Ising model for certain parameter values of the corresponding interactions, as well as to the topological entropy of certain nearest-neighbor \mathbb{Z}^2 shifts of finite type, generalizing a result in [R. Pavlov. Approximating the hard square entropy constant with probabilistic methods. *Ann. Probab.* to appear].

1. Introduction

The concept of entropy is fundamental to the study of dynamical systems both in topological dynamics, where it arises as topological entropy for continuous maps, and in ergodic theory, where it arises as measure-theoretic entropy for measure-preserving transformations.

Of particular interest in symbolic dynamics are dynamical systems known as shifts of finite type. We restrict our attention to nearest neighbor \mathbb{Z}^d shifts of finite type (nearest neighbor \mathbb{Z}^d -SFT); such an SFT X is specified by a finite alphabet \mathcal{A} and a set of translation-invariant adjacency rules: X is the subset of $\mathcal{A}^{\mathbb{Z}^d}$ of all configurations on \mathbb{Z}^d which satisfy the adjacency rules. Here, the underlying dynamics are given by the group of translations by vectors in \mathbb{Z}^d . The topological entropy $h(X)$ is defined as the asymptotic growth rate of the number of configurations on finite rectangles that extend to elements of X (more precise definitions for nearest neighbor \mathbb{Z}^d -SFT, topological entropy and other concepts used in this introduction are given in §2).

The most prominent non-trivial example in dimension $d = 1$ (i.e. nearest neighbor \mathbb{Z} -SFT) is the golden mean shift, defined as the set of all bi-infinite 0–1 sequences that do not contain two adjacent 1s. Its two-dimensional analogue, known as the *hard square shift* \mathcal{H} , is defined as the set of all 0–1 configurations on \mathbb{Z}^2 such that 1s are never adjacent horizontally or vertically.

The topological entropy of a nearest neighbor \mathbb{Z} -SFT X is easy to compute: namely, $h(X)$ is the logarithm of the largest eigenvalue of a non-negative integer matrix defined by the restricted adjacency rules. The topological entropy of the golden mean shift turns out to be the log of the golden mean (hence the name for this SFT). However, it is very difficult in general to compute the topological entropy of a nearest neighbor \mathbb{Z}^2 -SFT, and exact values are known in only a handful of cases. Even for the hard square shift, the topological entropy is not known.

Given a nearest neighbor \mathbb{Z}^2 -SFT X , the allowed configurations on a strip of height n form what is effectively a nearest neighbor \mathbb{Z} -SFT X_n ; here, the alphabet consists of columns of height n that obey the vertical adjacency rules, with two adjacent columns required to satisfy the horizontal adjacency rules. The sequence of differences $h(X_{n+1}) - h(X_n)$ does not always converge to $h(X)$ (see [Pi] and [P]). However, in [P], Pavlov proved that for the hard square shift $X = \mathcal{H}$, not only does this sequence converge to $h(X)$, but it does so exponentially fast (as a function of n). While this does not give an exact expression for $h(X)$, it does show that $h(X)$ can be approximated relatively well. In particular, a consequence of the approximation result from [P] is that there is a polynomial time algorithm which on input k produces an estimate of $h(\mathcal{H})$ guaranteed to be accurate within $1/k$. This is in stark contrast to the main result from [HM], which implies that there exist numbers which occur as the topological entropy of a nearest neighbor \mathbb{Z}^2 -SFT and are arbitrarily poorly computable.

While topological entropy can be viewed as a purely combinatorial object, the proof in [P] uses measure-theoretic tools. For a translation-invariant measure μ on $\mathcal{A}^{\mathbb{Z}^2}$, there is an analogous notion of measure-theoretic entropy $h(\mu)$. If the support of μ is contained in a nearest neighbor \mathbb{Z}^2 -SFT X , then $h(\mu) \leq h(X)$ and there is always at least one measure μ such that $h(\mu) = h(X)$. For $X = \mathcal{H}$, there is a unique measure of maximal entropy μ_{\max} . There is also a unique measure of maximal entropy μ_n for each \mathcal{H}_n . Using results on ‘disagreement percolation’ from [BS], it was shown that the sequence of differences $h(\mu_{n+1}) - h(\mu_n)$ converges exponentially fast to $h(\mu_{\max})$, and one concludes that $h(\mathcal{H}_{n+1}) - h(\mathcal{H}_n)$ converges exponentially fast to $h(\mathcal{H})$, as desired.

Of critical importance to the proof is the fact that μ_{\max} is a \mathbb{Z}^2 -Markov random field (MRF), which is, roughly speaking, a measure on $\mathcal{A}^{\mathbb{Z}^2}$ such that for any finite subset $S \subset \mathbb{Z}^2$ the conditional probability distribution on configurations on S given a configuration u on $\mathbb{Z}^2 \setminus S$ depends only on the restriction of u to the boundary of S . For μ_{\max} , these conditional probabilities are uniform over allowed configurations on S given u .

In this paper, we generalize the main result of [P] to more general translation-invariant \mathbb{Z}^2 -MRFs μ , with non-uniform conditional probabilities. Using results from [BM], instead of [BS], we give, in §3, a measure-theoretic analogue for certain \mathbb{Z}^2 -MRFs. Namely, our Theorem 3.22 asserts that for sufficiently large n , the MRF μ induces MRFs μ_n on strips of height n , with appropriate boundary conditions, such that the sequence of

differences $h(\mu_{n+1}) - h(\mu_n)$ converges exponentially fast to $h(\mu)$ (here, μ_n is viewed as a one-dimensional stationary process on sequences of configurations of n -high columns); this result requires the existence of suitable boundary rows for sufficiently large n and a condition on the probability distributions on configurations at a site in \mathbb{Z}^2 , conditioned on configurations on its four nearest neighbors. The condition (which is from [BM]) is that any two such (conditional) probability distributions should not be too different—more precisely, the total variation distance between any such distributions should be less than the critical value for site percolation in \mathbb{Z}^2 . This condition is similar in spirit to the classical Dobrushin uniqueness criterion. The induced MRF μ_n is defined by restricting the conditional probability specifications of μ to the strip of height n , with appropriate boundary conditions imposed on the row immediately above the top row of the strip and the row immediately below the bottom row of the strip. We note, in particular, that μ_n is *not* the usual marginalization of μ to the strip; this latter process is typically not even an MRF.

This all becomes more concrete when the MRF μ is a Gibbs state for a nearest neighbor interaction Φ on a nearest neighbor \mathbb{Z}^2 -SFT X which satisfies a strong irreducibility condition (Gibbs states are discussed in §4 and the strong irreducibility condition and consequences are discussed in §5). In this case, the induced MRFs μ_n are translation-invariant first-order Markov chains whose transition probabilities are easily computed from the interaction (Proposition 6.3 in §6). There is a simple closed form for the entropy of such a Markov chain, which in spirit is similar to the closed form for topological entropy of a nearest neighbor \mathbb{Z} -SFT.

The interaction Φ defines a continuous function f_Φ on X . The pressure $P_X(f_\Phi)$ of such a function is defined as the asymptotic growth rate of arrays which are, roughly speaking, weighted by $\exp(f_\Phi)$. Using an equivalent variational formula for $P_X(f_\Phi)$, given in terms of translation-invariant measures supported within X , one can apply Theorem 3.22 to obtain exponentially fast approximations to pressures of such functions f_Φ by differences of pressures of induced functions on strips of height n (Theorem 7.2); these pressures are computed as largest eigenvalues of explicit matrices. A corollary of this result is Theorem 8.1, which expresses the computability of $P_X(f_\Phi)$ in terms of computability of the values of Φ .

Finally, examples of Gibbs states and corresponding pressures are given in §9. We consider two classical examples: the two-dimensional hard core model, given by an interaction parameterized by activity level a , and the two-dimensional Ising antiferromagnet model, parameterized by inverse temperature β and external field h . Explicit ranges of values of these parameters are given for which Theorem 7.2 applies.

When $\Phi = 0$, then $f_\Phi = 0$ and $P_X(f_\Phi)$ reduces to $h(X)$. It follows that the approximation result for pressure (Theorem 7.2) can be used to obtain exponentially fast approximations to $h(X)$ for certain nearest neighbor \mathbb{Z}^2 -SFTs. In particular, this result recovers the main result of [P] and extends that result to other nearest neighbor \mathbb{Z}^2 -SFTs, examples of which are given in §9.

After this paper was submitted, we were informed of related work by Gamarnik and Katz [GK] who introduced a sequence of explicit upper and lower bounds that approximate pressure to within $1/k$ in time polynomial in k , specifically for a wide range of parameter

values for the hard square and monomer-dimer model in any dimension, but can be extended to many other models. Their work was based on methods developed in computer science for efficiently counting the number of independent sets in finite graphs [Wei].

The techniques of [GK] are related to those of our paper. In particular, both make use of the concept of correlation decay: configurations on sets far away from a given site have little effect on the configuration at that site. Both their work and ours was, in part, motivated by the observation that the more traditional approach which computes normalized sequence of strip pressures, based on transfer matrices, appears to converge slowly (see [GK, §3.3], which refers to Simon [Sim], and comments in the introduction of [P]).

Yet the sequences of approximations are different: the approximations in [GK] are given by approximating a single conditional probability p , via a recursion for which p is a fixed point; ours is a sequence of successive differences of strip pressures and makes explicit use of transfer matrices (differences of strip pressures rather than normalized strip pressure). While we have restricted attention to only two dimensional models, it appears that the papers apply to different, but intersecting, classes of parameter values and models.

2. Definitions and preliminaries

An undirected graph G consists of a set of vertices (or sites) $V(G)$ and a set of edges (or nearest neighbors) $E(G)$ of (unordered) pairs of distinct vertices. All graphs we consider will be countable and locally finite. Two vertices $v, w \in V(G)$ are said to be adjacent if $\{v, w\} \in E(G)$. For finite sets $U_1, U_2 \subset V(G)$, let $E(U_1, U_2)$ denote the set of all edges in G with one vertex in U_1 and the other in U_2 .

For any $d > 0$, we (in a slight abuse of notation) use \mathbb{Z}^d to denote the d -dimensional cubic lattice, the graph defined by

$$V(\mathbb{Z}^d) = \mathbb{Z}^d \quad \text{and} \quad E(\mathbb{Z}^d) = \left\{ \{u, v\} : \sum_{i=1}^d |u_i - v_i| = 1 \right\}.$$

The boundary of a set $S \subset V(G)$ within a graph G , which is denoted by $\partial(S, G)$, is the set of $v \in V(G) \setminus S$ which are adjacent to some element of S . If we refer to simply the boundary of a set S , or write ∂S , then the graph G is assumed to be \mathbb{Z}^2 . In the case where S is a singleton $\{v\}$, we call the boundary the set of neighbors N_v^G , which is just the set of $w \in V(G)$ adjacent to v . Again, when no mention of G is made, it is assumed to be \mathbb{Z}^2 .

For any integers $a < b$, we use $[a, b]$ to denote $\{a, a + 1, \dots, b\}$.

An alphabet \mathcal{A} is a finite set with at least two elements.

A configuration u on the alphabet \mathcal{A} in the graph G is any mapping from a non-empty subset S of $V(G)$ to \mathcal{A} , where S is called the shape of u . For any configuration u with shape S and any $T \subseteq S$, denote by $u|_T$ the restriction of u to T , i.e. the subconfiguration of u occupying T . For S, T disjoint sets, $x \in \mathcal{A}^S$ and $y \in \mathcal{A}^T$, xy denotes the configuration on $S \cup T$ defined by $(xy)|_S = x$ and $(xy)|_T = y$, which we call the concatenation of x and y . We will sometimes informally identify a configuration x on a shape S with the corresponding configuration on a translate $S + v$, namely the configuration y on $S + v$ defined by $y_u = x_{u-v}$.

For any d , we use σ to denote the natural shift action on $\mathcal{A}^{\mathbb{Z}^d}$ defined by $(\sigma_v(x))(u) = x(u + v)$.

For any alphabet \mathcal{A} and graph G , $\mathcal{A}^{V(G)}$ is a topological space when endowed with the product topology (where \mathcal{A} has the discrete topology), and any subsets will inherit the induced topology. We will also frequently speak of measures on $\mathcal{A}^{V(G)}$, and all such measures in this paper will be Borel probability measures. This means that any μ is determined by its values on the sets $[w] := \{x \in \mathcal{A}^{V(G)} : x|_S = w\}$, where w is a configuration with arbitrary finite shape $S \subseteq V(G)$. Such sets are called *cylinder sets*, and for notational convenience, rather than referring to a cylinder set $[w]$ within a measure or conditional measure, we just use the configuration w . For instance, $\mu(w \cap v|u)$ represents the conditional measure $\mu([w] \cap [v]|[u])$.

A measure μ on $\mathcal{A}^{\mathbb{Z}^d}$ is *translation-invariant* (or *stationary*) if $\mu(A) = \mu(\sigma_v A)$ for all measurable sets A and $v \in \mathbb{Z}^d$. A translation-invariant measure μ on $\mathcal{A}^{\mathbb{Z}^d}$ is *ergodic* if whenever $U \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is measurable and translation-invariant, then $\mu(U) = 0$ or 1 .

Let d be a positive integer. Let $\mathcal{E}_1, \dots, \mathcal{E}_d \subseteq \mathcal{A}^2$. The *nearest neighbor \mathbb{Z}^d shift of finite type (nearest neighbor \mathbb{Z}^d -SFT) X* , defined by $\mathcal{E}_1, \dots, \mathcal{E}_d$, is the set X of all $x \in \mathcal{A}^{\mathbb{Z}^d}$ such that whenever $u \in \mathbb{Z}^d$ and $1 \leq i \leq d$, we have $x(u)x(u + e_i) \in \mathcal{E}_i$, where e_i is the i th standard basis vector. We say that X is a nearest neighbor SFT if it is a nearest neighbor \mathbb{Z}^d -SFT for some d .

When $d = 1$, we write $\mathcal{E} = \mathcal{E}_1$. Any nearest neighbor \mathbb{Z} -SFT X defined by \mathcal{E} has an associated square $|\mathcal{A}| \times |\mathcal{A}|$ matrix A , called the *adjacency matrix*, which is defined by $a_{i,j} = \chi_{\mathcal{E}}(i, j)$. In other words, $a_{i,j} = 1$ if and only if $(i, j) \in \mathcal{E}$.

The *language* of X is

$$\mathcal{L}(X) = \bigcup_{\{S \subseteq \mathbb{Z}^d, |S| < \infty\}} \mathcal{L}_S(X)$$

where

$$\mathcal{L}_S(X) = \{x|_S : x \in X\}.$$

For a subset $S \subseteq \mathbb{Z}^d = V(\mathbb{Z}^d)$, finite or infinite, a configuration $x \in \mathcal{A}^S$ is *globally admissible* for X if x extends to a configuration on all of \mathbb{Z}^d . So, the language $\mathcal{L}(X)$ is precisely the set of globally admissible configurations on finite sets.

A configuration $x \in \mathcal{A}^S$ is *locally admissible* for X if for all edges $e = \{u, u + e_i\}$ contained in S we have $x|_e \in \mathcal{E}_i$. We note that technically there is an ambiguity here since several choices for \mathcal{E}_i could induce the same nearest neighbor SFT. For this reason, we will always think of a nearest neighbor SFT as being ‘equipped’ with a specific choice of the sets \mathcal{E}_i .

Example 2.1. The \mathbb{Z}^2 hard square shift \mathcal{H} is the nearest neighbor shift of finite type with alphabet $\{0, 1\}$ and $\mathcal{E}_1 = \mathcal{E}_2 = \{(0, 0), (0, 1), (1, 0)\}$.

Given any measure μ on $\mathcal{A}^{\mathbb{Z}^d}$ and any rectangular prism $R = \prod [1, n_i]$, we can associate a *R-higher power code* $\mu^{[R]}$ of μ , defined as the image of μ under the mapping $\phi_R : \mathcal{A}^{\mathbb{Z}^d} \rightarrow (\mathcal{A}^R)^{\mathbb{Z}^d}$ defined by $(\phi_R x)(v_1, \dots, v_d) = x|_{\prod [n_i v_i, n_i(v_i+1)-1]}$. For any nearest neighbor \mathbb{Z}^d -SFT X , $\phi_R(X)$ is also a nearest neighbor \mathbb{Z}^d -SFT, which we call the *R-higher power code* of X and denote by $X^{[R]}$.

We define a square non-negative matrix A to be *primitive* if some power A^n has all positive entries. This allows us to define the notion of mixing for two types of one-dimensional dynamical systems. A nearest neighbor \mathbb{Z} -SFT X is called *mixing* if and

only if its adjacency matrix (after discarding any letters of \mathcal{A} which do not actually appear in X) is primitive, and a Markov chain is called *mixing* if its transition probability matrix is primitive.

For any translation-invariant measure μ on $\mathcal{A}^{\mathbb{Z}^d}$, we may define its entropy as follows.

Definition 2.2. The *measure-theoretic entropy* of a translation-invariant measure μ on $\mathcal{A}^{\mathbb{Z}^d}$ is defined by

$$h(\mu) = \lim_{j_1, j_2, \dots, j_d \rightarrow \infty} \frac{-1}{j_1 j_2 \cdots j_d} \sum_{w \in \mathcal{A}^{\prod_{i=1}^d [1, j_i]}} \mu(w) \log(\mu(w)),$$

where terms with $\mu(w) = 0$ are omitted.

We will also deal with measure-theoretic conditional entropy in this paper. It can be defined more generally, but for our purposes, we will define it only for a measure on $\mathcal{A}^{\mathbb{Z}}$ and specific type of partition of $\mathcal{A}^{\mathbb{Z}}$. For any partition ξ of a set S , and for any $s \in S$, we use $\xi(s)$ to denote the element of ξ which s is in. If ξ is a partition of an alphabet \mathcal{A} , then ϕ_ξ is the map on $\mathcal{A}^{\mathbb{Z}}$ defined by $\phi_\xi(x) = \cdots \xi(x_{-1})\xi(x_0)\xi(x_1) \cdots$.

We note that for any measure μ on $\mathcal{A}^{\mathbb{Z}}$ and any partition ξ of \mathcal{A} , the push-forward $\phi_\xi(\mu)$ of μ under the map ϕ_ξ is a measure on $\xi^{\mathbb{Z}}$.

Definition 2.3. For any translation-invariant measure μ on $\mathcal{A}^{\mathbb{Z}}$ and any partition ξ of \mathcal{A} , the conditional measure-theoretic entropy of μ with respect to ξ is

$$h(\mu|\xi) = \lim_{k \rightarrow \infty} \frac{-1}{2k + 1} \sum_{w \in \mathcal{A}^{[-k, k]}} \mu(w) \log \left(\frac{\mu(w)}{(\phi_\xi(\mu))(\xi(w_{-k}) \cdots \xi(w_k))} \right),$$

where again terms with $\mu(w) = 0$ are omitted.

Measure-theoretic conditional entropy is most useful because of the following decomposition formula. For a proof, see [P].

PROPOSITION 2.4. For any translation-invariant measure μ on $\mathcal{A}^{\mathbb{Z}}$, and any partition ξ of \mathcal{A} ,

$$h(\mu|\xi) = h(\mu) - h(\phi_\xi(\mu)).$$

The *weak topology* on the space of measures on $\mathcal{A}^{\mathbb{Z}}$ is the weakest topology under which the map $\mu \rightarrow \int f d\mu$ is continuous for every continuous function f on $\mathcal{A}^{\mathbb{Z}}$. Measure-theoretic (and conditional measure-theoretic) entropy is not continuous in the weak topology (though it is upper semicontinuous); see [Wal]. For this reason, we need to define the \bar{d} metric for measures, with respect to which the entropy map $\mu \mapsto h(\mu)$ is continuous and in fact Hölder. We first need the preliminary definition of a coupling.

Definition 2.5. For any measures μ on X and ν on Y , a coupling of μ and ν is a measure λ on $X \times Y$ for which $\lambda(A \times Y) = \mu(A)$ for any μ -measurable $A \subseteq X$ and $\lambda(X \times B) = \nu(B)$ for any ν -measurable $B \subseteq Y$. The set of couplings of μ and ν is denoted by $C(\mu, \nu)$.

Definition 2.6. For any measures μ and μ' on $\mathcal{A}^{\mathbb{Z}}$,

$$\bar{d}(\mu, \mu') = \limsup_{n \rightarrow \infty} \min_{\lambda \in C(\mu|_{[-n, n]}, \mu'|_{[-n, n]})} \int d_{2n+1}(x, y) d\lambda,$$

where d_k is the normalized k -letter Hamming distance between k -letter configurations given by $d_k(u, v) = 1/k \sum_{1 \leq i \leq k} (1 - \delta_{u(i)v(i)})$.

We briefly summarize some important properties of the \bar{d} distance (for more information, see [Rud] or [S]). We are interested only in the \bar{d} metric on the space of translation-invariant measures on $\mathcal{A}^{\mathbb{Z}}$ for a fixed alphabet \mathcal{A} . The \bar{d} metric is complete and dominates distribution distance in the sense that for any configuration w on a finite interval of length m , $|\mu(w) - \mu'(w)| \leq m\bar{d}(\mu, \mu')$. The following estimate implies that entropy is Hölder continuous of any exponent <1 , with respect to \bar{d} (see [Rud, Theorem 7.9] for a proof of the estimate in the ergodic case; the same estimate holds in the general translation-invariant case): letting $\epsilon = \bar{d}(\mu, \mu')$,

$$|h(\mu) - h(\mu')| \leq \epsilon \log |\mathcal{A}| - \epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon).$$

Finally, we define the topological pressure of a continuous function on a nearest neighbor SFT, following [Rue].

Let X be a nearest neighbor \mathbb{Z}^d -SFT, and let $f : X \rightarrow \mathbb{R}$ be a continuous function. The topological pressure of f on X can be defined in several ways; one is as the purely topological notion of the asymptotic growth rate of the number of (locally or globally) admissible arrays in X , ‘weighted by f ’. For our purposes though, the following definition, which is a consequence of the variational principle (see [Mi] for a short proof), is more convenient.

Definition 2.7. Given a nearest neighbor \mathbb{Z}^d -SFT X and $f \in C(X)$, the (topological) pressure of f on X is

$$P(f) = P_X(f) = \sup_{\mu} \left(h(\mu) + \int f d\mu \right),$$

where the supremum is taken over all translation-invariant measures μ supported on X .

The supremum is always achieved and any measure which achieves the supremum is called an *equilibrium state* for X and f [Wal].

In the special case when $f = 0$, $P(f)$ is called the *topological entropy* $h(X)$ of X , and any equilibrium state is called a *measure of maximal entropy* for X .

3. Exponential approximation of MRF entropies

The main measures we will study in this paper are Markov random fields (or MRFs) on sets of configurations on a graph G .

Definition 3.1. For any graph G and finite alphabet \mathcal{A} , a measure μ on $\mathcal{A}^{V(G)}$ is called a G -Markov random field (or G -MRF) if, for any finite $S \subset V(G)$, any $\eta \in \mathcal{A}^S$, any finite $T \subset V(G)$ such that $\partial(S, G) \subseteq T \subseteq V(G) \setminus S$, and any $\delta \in \mathcal{A}^T$ with $\mu(\delta) \neq 0$,

$$\mu(\eta | \delta|_{\partial(S,G)}) = \mu(\eta|\delta).$$

Informally, μ is a G -MRF if, for any finite $S \subset V(G)$, the sites in S and the sites in $V(G) \setminus (S \cup \partial(S, G))$ are μ -conditionally independent given the sites on $\partial(S, G)$.

We will sometimes refer to a G -MRF simply as an MRF when G is clear from context. We note that our definition of MRF differs slightly from the usual one, where the right-hand

side would involve conditioning on an entire configuration on $V(G) \setminus S$ almost everywhere rather than arbitrarily large finite subconfigurations of it. However, the definitions are equivalent and the finite approach leads to simpler calculations and proofs.

Definition 3.2. For a graph G and finite alphabet \mathcal{A} , a G -specification Λ is defined by a set of finitely supported probability measures

$$\{\Lambda^\delta(\cdot) \mid S \subset V(G), |S| < \infty, \delta \in \mathcal{A}^{\partial(S,G)}\},$$

where, for each $\Lambda^\delta(\cdot)$, \cdot ranges over all configurations in \mathcal{A}^S .

Technically, the notation Λ^δ should include the set S , because for general graphs, a configuration on ∂S need not determine S . However, in this paper we only consider graphs for which ∂S determines S (namely, \mathbb{Z}^d and the strip subgraphs $H_{m,n}$ defined below), and so we use the simpler notation.

Again we will sometimes refer to a G -specification simply as a specification. We say that a \mathbb{Z}^d -specification is *translation-invariant* if $\Lambda^{\sigma_v \delta} = \sigma_v \Lambda^\delta$ for all δ and $v \in \mathbb{Z}^d$.

Definition 3.3. For any graph G , μ a measure on $\mathcal{A}^{V(G)}$, any finite set $S \subset V(G)$, and any $\delta \in \mathcal{A}^{\partial(S,G)}$ with $\mu(\delta) > 0$, denote by μ^δ the measure on \mathcal{A}^S defined by $\mu^\delta(u) = \mu(u|\delta)$.

Definition 3.4. For any graph G and finite alphabet \mathcal{A} , and G -specification Λ , a G -MRF μ is associated to Λ if $\mu^\delta = \Lambda^\delta$ for all finite $S \subseteq V(G)$ and $\delta \in \mathcal{A}^{\partial(S,G)}$ with $\mu(\delta) > 0$.

Note that in checking whether an MRF μ is associated to a specification Λ , many of the Λ^δ are totally irrelevant; namely those which correspond to δ which have zero μ -measure.

We say that a G -specification Λ is *valid* if there is at least one G -MRF associated to it. If there is exactly one such MRF, we denote it by $\mu(\Lambda)$.

Often a specification is required to satisfy a consistency condition; see [Ge, Definition 1.23]. This condition is important for results that assert the existence of an MRF associated to a given specification: the existence of an MRF μ forces certain consistencies of specifications on the support of μ . However, in our work, we do not need to require consistency: whenever we need existence, we will either assume it (i.e., that the specification is valid) or assume a condition that guarantees it (for instance, in Proposition 3.13). And the consistency condition is not needed in uniqueness results such as Theorem 3.9.

To obtain good approximations of MRF entropies, we will use a condition from [BM] on G -specifications, which was used there to prove uniqueness of the associated MRF. We first need some definitions.

Definition 3.5. For any finite set S and two measures μ and ν on S , the variational distance between μ and ν is

$$d(\mu, \nu) = \frac{1}{2} \sum_{s \in S} |\mu(s) - \nu(s)|.$$

We note that $d(\mu, \nu) = 1$ if and only if μ and ν have disjoint supports.

Definition 3.6. For any graph G , finite alphabet \mathcal{A} , $g \in V(G)$, and valid G -specification Λ , define

$$q_g(\Lambda) := \max_{\delta, \delta' \in \mathcal{A}^{N_g^G}} d(\Lambda^\delta, \Lambda^{\delta'})$$

and $q(\Lambda) := \sup_{g \in V(G)} q_g(\Lambda)$.

Definition 3.7. For any finite alphabet \mathcal{A} , graph G , and probability distribution λ on \mathcal{A} , P_λ represents the Bernoulli (independent and identically distributed) measure on $\mathcal{A}^{V(G)}$ whose distribution on each site is λ .

Definition 3.8. For any graph G , the *critical probability for site percolation on G* , denoted by $p_c(G)$, is defined as the supremum of $q \in [0, 1]$ for which, given the alphabet $\{0, 1\}$ and the graph G , the $P_{(1-q, q)}$ -probability that there is an infinite connected subgraph of G with 1s at every site is zero.

We point out that percolation theory is an extremely rich area of mathematics, which we give short shrift to here. For more information, see [Gr].

When the graph G is omitted, we understand p_c to denote $p_c(\mathbb{Z}^2)$. Simulations suggest that $p_c \approx 0.593$, but the best-known lower bound is $p_c > 0.556$, proved by van den Berg and Ermakov [BE].

THEOREM 3.9. [BM, Corollary 2] *If Λ is a valid G -specification and $q(\Lambda) < p_c(G)$, then there is a unique G -MRF associated to Λ .*

Theorem 3.9 is, roughly speaking, proved by showing that for a G -specification Λ with $q(\Lambda) < p_c(G)$, boundary conditions on large sets (such as rectangles in \mathbb{Z}^2) exert very little influence on sites near the center. It will be necessary for us to quantify exactly how this influence decays, and so we will use the methods of [BM] to prove some finitistic results.

THEOREM 3.10. *For any valid \mathbb{Z}^2 -specification Λ with $q(\Lambda) < p_c$ (with unique associated MRF $\mu = \mu(\Lambda)$), there exist $K, L > 0$ such that for any non-empty finite set $S \subset \mathbb{Z}^2$, for any rectangle $R \supset S$, and for any configurations δ and δ' on ∂R with positive $\mu(\Lambda)$ -probability, there exists $\lambda \in C(\mu^\delta|_S, \mu^{\delta'}|_S)$ such that, for any $s \in S$, $\lambda(\{(x, y) : x(s) \neq y(s)\}) \leq Ke^{-Ld}$, where d is the distance between S and the set of $t \in \partial R$ for which $\delta(t) \neq \delta'(t)$. (We take $d = \infty$ if the latter set is empty.)*

Proof. Given any such Λ , S , R , δ , and δ' , [BM, Theorem 1] proves the existence of $\lambda' \in C(\mu^\delta, \mu^{\delta'})$ with the following two properties. (Following [BM], we will sometimes think of λ' as a measure on $\mathcal{A}^{R \cup \partial R} \times \mathcal{A}^{R \cup \partial R}$, where pairs (u, u') in the support of λ' are thought of as equivalent to $(u\delta, u'\delta')$.)

(i) Define the map ϕ from $(\mathcal{A}^{R \cup \partial R}) \times (\mathcal{A}^{R \cup \partial R})$ to $\{0, 1\}^{R \cup \partial R}$ by $(\phi(x, y))(v) = 1$ if and only if $x(v) \neq y(v)$. Then the measure $\phi\lambda'$ on $\{0, 1\}^{R \cup \partial R}$ is stochastically dominated by $P_{(1-q, q)}$, where $q = q(\Lambda)$. (We do not define stochastic dominance in general, but can give a simple definition which suffices for our setup. Given a finite set S and measures μ, ν on $\{0, 1\}^S$, μ is stochastically dominated by ν if for any set C of configurations which is closed under changing zeros to ones, $\mu(C) \leq \nu(C)$.)

(ii) For a set of $(x, y) \in (\mathcal{A}^{R \cup \partial R}) \times (\mathcal{A}^{R \cup \partial R})$ with λ' -probability 1 and for any $v \in R$, $x(v) \neq y(v)$ if and only if there is a path P of sites in \mathbb{Z}^2 from v to ∂R for which $x(p) \neq y(p)$ for all $p \in P$.

Note that for any fixed $v \in R$, this means that

$$\begin{aligned} &\lambda'(\{(x, y) : x(v) \neq y(v)\}) \\ &= \lambda'(\{(x, y) : \text{there is a path } P \text{ from } v \text{ to } \partial R \text{ such that } x(p) \neq y(p) \forall p \in P\}) \\ &= (\phi\lambda')(\{\text{there is a path } P \text{ of 1s from } v \text{ to } \partial R\}) \\ &\leq P_{(1-q,q)}(\{\text{there is a path of 1s from } v \text{ to } \partial R\}). \end{aligned}$$

A classical theorem proved by Menshikov [Me] and Aizenmann and Barsky [AB] shows that for any $q < p_c$, there exist $K = K(q)$ and $L = L(q)$ so that for any n , $P_{(1-q,q)}$ (there is a path of 1s from 0 to $\partial[-n, n]^2$) $< Ke^{-Ln}$. This clearly implies that $P_{(1-q,q)}$ (there is a path of 1s from v to ∂R) $< Ke^{-Ld_v}$, where d_v is the distance from v to the set of sites in ∂R at which δ and δ' disagree.

Therefore, if we define $\lambda = \lambda'|_{(\mathcal{A}^S \times \mathcal{A}^S)}$, then for any $s \in S$, $\lambda(\{(x, y) : x(s) \neq y(s)\}) = \lambda'(\{(x, y) : x(s) \neq y(s)\}) \leq Ke^{-Ld_s} \leq Ke^{-Ld}$, where d is just the minimum value of d_s for $s \in S$. □

We will prove a slightly more general version of Theorem 3.10 for δ and δ' on the boundaries of possibly different rectangles.

THEOREM 3.11. *For any valid \mathbb{Z}^2 -specification Λ with $q(\Lambda) < p_c$ (with unique associated MRF $\mu = \mu(\Lambda)$), there exist $K, L > 0$ such that for any finite set $S \subset \mathbb{Z}^2$, for any rectangles $R' \supset R \supset S$, and for any configurations δ and δ' on ∂R and $\partial R'$ respectively with positive $\mu(\Lambda)$ -probability, there exists $\lambda \in C(\mu^\delta|_S, \mu^{\delta'}|_S)$ such that for any $s \in S$, $\lambda(\{(x, y) : x(s) \neq y(s)\}) < Ke^{-Ld}$, where d is the distance between S and the set of $t \in \partial R$ for which either $t \notin \partial R'$ or $t \in \partial R'$ and $\delta(t) \neq \delta'(t)$.*

Proof. We will be using Theorem 3.10. First, let's note that $\mu^{\delta'}|_R$ can be written as a weighted average of the measures μ^η , where η ranges over all configurations on ∂R which agree with δ' on $\partial R \cap \partial R'$. This means that, in particular, $\mu^{\delta'}|_S = \sum_\eta \alpha_\eta (\mu^\eta|_S)$ for some non-negative numbers α_η summing to 1. By Theorem 3.10 there exist $K, L > 0$ so that for any η there exists $\lambda_\eta \in C(\mu^\delta|_S, \mu^\eta|_S)$ with the property that for any $s \in S$, $\lambda_\eta(\{(x, y) : x(s) \neq y(s)\}) < Ke^{-Ld_\eta}$, where d_η is the distance from any $s \in S$ to the set of $t \in \partial R$ for which $\delta(t) \neq \eta(t)$. Take $\lambda = \sum_\eta \alpha_\eta \lambda_\eta$. It is clear that $\lambda \in C(\mu^\delta|_S, \mu^{\delta'}|_S)$. Also, clearly

$$\lambda(\{(x, y) : x(s) \neq y(s)\}) = \sum_\eta \alpha_\eta \lambda_\eta(\{(x, y) : x(s) \neq y(s)\}) \leq \sum_\eta \alpha_\eta Ke^{-Ld_\eta}.$$

Note that for any η agreeing with δ' on $\partial R \cap \partial R'$, $d_\eta \geq d$ as defined in the theorem. Therefore, clearly $\lambda(\{(x, y) : x(s) \neq y(s)\}) < Ke^{-Ld}$, and we are done. □

Given a valid \mathbb{Z}^2 -specification Λ with $q(\Lambda) < p_c$, we wish to define specifications (and associated MRFs) on the maximal subgraphs $H_{m,n}$ of \mathbb{Z}^2 induced by $V(H_{m,n}) = \mathbb{Z} \times [m, n]$, yielding measures on sets of configurations on bi-infinite horizontal strips.

Of course Λ itself does not contain enough information to define an $H_{m,n}$ -specification; there exist sets of the form $\partial(S, H_{m,n})$ for some finite set $S \subset H_{m,n}$, but which are not

expressible as ∂T for any finite set $T \subset \mathbb{Z}^2$. (For instance, picture the ‘three-sided’ boundary of a rectangle which includes part of the top row in $H_{m,n}$.) We therefore supplement Λ with boundary conditions as follows. Suppose that $t, b \in \mathcal{A}^{\mathbb{Z}}$, and we will define a $H_{m,n}$ -specification $\Lambda_{m,n,t,b}$.

For any finite $S \subset \mathbb{Z} \times [m, n]$, $\eta \in \mathcal{A}^S$, and $\delta \in \mathcal{A}^{\partial(S, H_{m,n})}$, define

$$\Lambda_{m,n,t,b}^\delta(\eta) = \Lambda^\xi(\eta), \tag{1}$$

where $\xi = (t|_{\partial S \cap (\mathbb{Z} \times \{n+1\})})(b|_{\partial S \cap (\mathbb{Z} \times \{m-1\})})\delta$, the concatenation of $t|_{\partial S \cap (\mathbb{Z} \times \{n+1\})}$, $b|_{\partial S \cap (\mathbb{Z} \times \{m-1\})}$, and δ . In other words, a configuration on $\partial(S, H_{m,n})$ is supplemented by symbols from t and b above and below $H_{m,n}$, if necessary, to extend it to ∂S . The following gives a sufficient condition on m, n, t, b for the validity of $\Lambda_{m,n,t,b}$.

Definition 3.12. For a \mathbb{Z}^2 -specification Λ with alphabet \mathcal{A} , integers $m < n$ and $t, b \in \mathcal{A}^{\mathbb{Z}}$, we say that m, n, t, b is compatible with Λ if there exists an MRF μ associated to Λ such that for all sufficiently large k there exists $\delta_k \in \mathcal{A}^{\partial([-k,k] \times [m,n])}$ with positive μ -measure whose top row is $t|_{[-k,k]}$ and whose bottom row is $b|_{[-k,k]}$.

PROPOSITION 3.13. *For a valid \mathbb{Z}^2 -specification Λ with alphabet \mathcal{A} , integers $m < n$ and $t, b \in \mathcal{A}^{\mathbb{Z}}$, if m, n, t, b is compatible with Λ , then $\Lambda_{m,n,t,b}$ is a valid $H_{m,n}$ -specification.*

Proof. Let μ be an associated MRF and for all sufficiently large k , δ_k as described in Definition 3.12. Define the measures μ^{δ_k} . We wish to take a weak limit of a subsequence of these measures, so they must be extended to measures on all of $\mathcal{A}^{\mathbb{Z} \times [m,n]}$: choose any $a \in \mathcal{A}$, and for each k extend each configuration in the support of μ^{δ_k} to all of $\mathbb{Z} \times [m, n]$ by filling all unoccupied sites with a s.

By definition of $\Lambda_{m,n,t,b}$ and the fact that μ is an MRF associated to Λ , any weak limit of a subsequence of the measures μ^{δ_k} is an MRF associated to the specification $\Lambda_{m,n,t,b}$, and so $\Lambda_{m,n,t,b}$ is valid. □

While the compatibility condition may be difficult to check in general, it is checkable for certain special kinds of MRFs introduced in later sections.

We can also give a sufficient condition for uniqueness of an $H_{m,n}$ -MRF associated to $\Lambda_{m,n,t,b}$. In fact, it requires a less restrictive bound on $q(\Lambda)$, which will be useful for some later discussions.

PROPOSITION 3.14. *For any integers $m < n$ and any valid $H_{m,n}$ -specification $\Lambda_{m,n,t,b}$ induced by a \mathbb{Z}^2 -specification Λ with $q(\Lambda) < 1$ and boundary conditions t and b , $\Lambda_{m,n,t,b}$ has a unique associated MRF.*

Proof. Consider any such $\Lambda, m, n, t,$ and b for which $\Lambda_{m,n,t,b}$ is a valid specification. For any $i \in V(H_{m,n}) = \mathbb{Z} \times [m, n]$, by the definition of $\Lambda_{m,n,t,b}$, $q_i(\Lambda_{m,n,t,b})$ and $q_i(\Lambda)$ are both maxima of $d(\Lambda^\delta, \Lambda^{\delta'})$ over sets of pairs $\delta, \delta' \in \mathcal{A}^{N_i}$. However, for $q_i(\Lambda)$, one maximizes over all such pairs, and for $q_i(\Lambda_{m,n,t,b})$, one may be maximizing over a smaller set. (For instance, if i is part of the top row of $H_{m,n}$, then one only considers configurations on the neighbors where the neighbor above i is equal to $t(i + (0, 1))$.) Therefore, $q_i(\Lambda_{m,n,t,b}) \leq q_i(\Lambda)$. Since i was arbitrary, $q(\Lambda_{m,n,t,b}) \leq q(\Lambda)$.

It now suffices to show that $p_c(H_{m,n}) = 1$; the proposition then follows from Theorem 3.9. For any $p < 1$, if sites of $H_{m,n}$ are independently taken to be open with

probability p and closed with probability $1 - p$, then the probability that an entire column $\{i\} \times [m, n]$ is closed is $(1 - p)^{n-m+1} > 0$. This means that with $P_{(1-p,p)}$ -probability 1, there exist closed columns arbitrarily far to the left and right, meaning that there are no infinite open connected clusters. Therefore, since $p < 1$ was arbitrary, $p_c(H_{m,n}) = 1$ and we are done. \square

Clearly, when the hypotheses of Proposition 3.14 are satisfied, $\mu(\Lambda_{m,n,t,b})$ can be thought of as a measure on the one-dimensional full shift $(\mathcal{A}^{[m,n]})^{\mathbb{Z}}$, and we will interpret $\mu(\Lambda_{m,n,t,b})$ in this way for further discussions about \bar{d} distance and entropy. It should always be clear from context which viewpoint is being used.

PROPOSITION 3.15. *For any valid \mathbb{Z}^2 -specification Λ with $q(\Lambda) < p_c$ (with unique associated MRF $\mu = \mu(\Lambda)$) and any $t, b \in \mathcal{A}^{\mathbb{Z}}$ for which there exists N such that $\mu(\Lambda_{-n,n,t,b})$ exists for all $n > N$, $\mu(\Lambda_{-n,n,t,b})$ approaches μ weakly as $n \rightarrow \infty$.*

Proof. By definition of $\Lambda_{-n,n,t,b}$, any weak limit of a subsequence of $\mu(\Lambda_{-n,n,t,b})$ is clearly a \mathbb{Z}^2 -MRF associated to Λ . (As before, we need to extend each $\mu(\Lambda_{-n,n,t,b})$ to a measure on all of $\mathcal{A}^{\mathbb{Z}^2}$; we do this by choosing any $a \in \mathcal{A}$ and extending each configuration in the support of $\mu(\Lambda_{-n,n,t,b})$ to all of \mathbb{Z}^2 by filling the unoccupied sites with a .) However, the only such MRF is μ . \square

We will now use Theorem 3.11 to derive couplings of marginalizations of $\mu(\Lambda_{1,n,t,b})$ and $\mu(\Lambda_{1,n+1,t,b})$ to substrips which imply their closeness in the \bar{d} metric.

THEOREM 3.16. *For any valid \mathbb{Z}^2 -specification Λ with $q(\Lambda) < p_c$ (with unique associated MRF $\mu = \mu(\Lambda)$), there exist $K, L > 0$ such that for any n and any $t, b \in \mathcal{A}^{\mathbb{Z}}$ such that $1, n, t, b$ and $1, n + 1, t, b$ are compatible with Λ , and for any $1 \leq i < i' \leq n$,*

$$\bar{d}(\mu(\Lambda_{1,n,t,b})|_{\mathbb{Z} \times [i,i'-1]}, \mu(\Lambda_{1,n+1,t,b})|_{\mathbb{Z} \times [i,i'-1]}) \leq (i' - i)Ke^{-L(n-i')}$$

and

$$\bar{d}(\mu(\Lambda_{1,n,t,b})|_{\mathbb{Z} \times [i,i'-1]}, \mu(\Lambda_{1,n+1,t,b})|_{\mathbb{Z} \times [i+1,i']}) \leq (i' - i)Ke^{-Li}.$$

Proof. We begin with the first inequality. For Λ as in the theorem, take the K and L guaranteed by Theorem 3.11. Fix t, b, n , and $i < i'$. For every sufficiently large k , take δ_k and δ'_k configurations on $\partial([-k, k] \times [1, n])$ and $\partial([-k, k] \times [1, n + 1])$ respectively with positive μ -measure which are both equal to $t|_{[-k,k]}$ on the top and $b|_{[-k,k]}$ on the bottom. Define $S_{i,i',k} = [-k, k] \times [i, i' - 1]$. By Theorem 3.11, for any $k > n - i'$ and $j < k - (n - i')$, there exists $\lambda_{i,i',j,k} \in C(\mu^{\delta_k}|_{S_{i,i',j}}, \mu^{\delta'_k}|_{S_{i,i',j}})$ for which the $\lambda_{i,i',j,k}$ -probability of disagreement at any site in $S_{i,i',j}$ is less than $Ke^{-L(n-i')}$. (This is because δ_k and δ'_k agree on their bottom row, and the distance from any site in $S_{i,i',j}$ to any other site in $\partial([-k, k] \times [1, n]) \cup \partial([-k, k] \times [1, n + 1])$ is at least $n - i'$.)

Now, for any fixed $i < i'$ and j , define $\lambda_{i,i',j}$ to be any weak limit of a subsequence of the couplings $\lambda_{i,i',j,k}$ as $k \rightarrow \infty$. Note that any weak subsequence of μ^{δ_k} approaches an MRF on $H_{1,n,t,b}$ associated to the specification $\Lambda_{1,n,t,b}$, and since $\mu(\Lambda_{1,n,t,b})$ is the unique such MRF, the sequence μ^{δ_k} itself must weakly approach $\mu(\Lambda_{1,n,t,b})$. Similarly, $\mu^{\delta'_k}$ weakly approaches $\mu(\Lambda_{1,n+1,t,b})$. Therefore,

$$\lambda_{i,i',j} \in C(\mu(\Lambda_{1,n,t,b})|_{S_{i,i',j}}, \mu(\Lambda_{1,n+1,t,b})|_{S_{i,i',j}}).$$

Also, it is clear that if we think of a configuration on $S_{i,i',j}$ as a $(2j + 1)$ -letter word on the alphabet of words on columns of height $i' - i$, then

$$E_{\lambda_{i,i',j}}(d_{2j+1}(x|_{[-j,j]}, y|_{[-j,j]})) < (i' - i)Ke^{-L(n-i')}.$$

Recalling the definition of \bar{d} , we then see that clearly

$$\bar{d}(\mu(\Lambda_{1,n,t,b})|_{\mathbb{Z} \times [i,i'-1]}, \mu(\Lambda_{1,n+1,t,b})|_{\mathbb{Z} \times [i,i'-1]}) \leq (i' - i)Ke^{-L(n-i')}.$$

To prove the second inequality, change the proof above by defining δ_k on $\partial([-k, k] \times [2, n + 1])$ instead. Then δ_k and δ'_k will agree on their top rows, and the rest of the proof goes through mostly unchanged. The distances from $\mathbb{Z} \times [i, i' - 1]$ and $\mathbb{Z} \times [i + 1, i']$ to the bottom rows of $H_{1,n}$ and $H_{1,n+1}$ respectively are at least i , which is why $n - i'$ is replaced by i . □

We will for now restrict our attention to translation-invariant \mathbb{Z}^2 -specifications Λ with $q(\Lambda) < p_c$ and constant boundary conditions $t, b \in \mathcal{A}^{\mathbb{Z}}$, which means that the measures $\mu(\Lambda_{m,n,t,b})$ (when they exist) will be translation-invariant as one-dimensional measures. (Otherwise, their horizontal shifts would also be MRFs associated to $\Lambda_{m,n,t,b}$, contradicting uniqueness of $\mu(\Lambda_{m,n,t,b})$.) We can then discuss the measure-theoretic entropies $h(\mu(\Lambda_{1,n,t,b}))$ and $h(\mu(\Lambda_{1,n+1,t,b}))$.

We will decompose these into conditional measure-theoretic entropies, and then use Theorem 3.16 and Hölder continuity of entropy (with respect to \bar{d}) to show that many of these entropies are exponentially close, finally showing that $h(\mu(\Lambda_{1,n+1,t,b})) - h(\mu(\Lambda_{1,n,t,b}))$ is exponentially close to $h(\mu)$. We first need some notation for special conditional measure-theoretic entropies.

For any a , define $R_a = \mathbb{Z} \times \{a\}$. For any $m < n$ and any interval $B \subseteq [m, n]$, we partition $\mathcal{A}^{[m,n]}$ by the letters appearing on B , and call this partition ξ_B . Then, for any translation-invariant measure μ on $(\mathcal{A}^{[m,n]})^{\mathbb{Z}}$ and disjoint adjacent intervals $B, C \subseteq [m, n]$, we make the notations

$$h_{\mu} \left(\bigcup_{b \in B} R_b \right) := h(\phi_{\xi_B}(\mu))$$

and

$$h_{\mu} \left(\bigcup_{c \in C} R_c \mid \bigcup_{b \in B} R_b \right) := h(\phi_{\xi_{B \cup C}}(\mu) | \xi_B).$$

Also, for any translation-invariant measure μ on $\mathcal{A}^{\mathbb{Z}^2}$, $h_{\mu}(\bigcup_{b \in B} R_b)$ will be understood to mean $h_{\mu|_{(\bigcup_{d \in D} R_d)}}(\bigcup_{b \in B} R_b)$ for any $D \supseteq B$; in other words, this expression is given meaning by marginalizing μ to any substrip containing the rows whose entropy is to be computed. It does not matter which D is used, since clearly this quantity depends only on the restriction of μ to $\bigcup_{b \in B} R_b$. We will interpret $h_{\mu}(\bigcup_{c \in C} R_c | \bigcup_{b \in B} R_b)$ in an analogous fashion.

The following is just a consequence of Proposition 2.4 for this new notation.

PROPOSITION 3.17. *For any $m < n$, μ a translation-invariant measure on $(\mathcal{A}^{[m,n]})^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{Z}^2}$, and B, C adjacent subintervals of $[m, n]$,*

$$h_{\mu} \left(\bigcup_{a \in B \cup C} R_a \right) = h_{\mu} \left(\bigcup_{b \in B} R_b \right) + h_{\mu} \left(\bigcup_{c \in C} R_c \mid \bigcup_{b \in B} R_b \right).$$

The following theorem can both be thought of as an extension of the Markov property in one dimension and a generalization of [P, Theorem 13].

THEOREM 3.18. *Let Λ be a translation-invariant \mathbb{Z}^2 -specification. Let n be a positive integer, $t, b \in \mathcal{A}^{\mathbb{Z}}$ be constant sequences and μ be an MRF associated to $\Lambda_{1,n,t,b}$. Then for any integers i and k with $1 \leq k < i \leq n$,*

$$h_\mu \left(R_i \left| \bigcup_{j=k}^{i-1} R_j \right. \right) = h_\mu(R_i | R_{i-1}).$$

Proof. We will prove the theorem for $k = 1$, and this suffices to prove the theorem for all $k < i$, since for any $1 \leq k < i$, conditioning on $\xi_{[k,i-1]}$ is an intermediate partition between $\xi_{[1,i-1]}$ and $\xi_{[i-1]}$. If the conditional entropies resulting from these two partitions are equal, then clearly any intermediate partition gives the same value. Fix any Λ, n, t, b , and i as in the statement of the theorem. For simplicity, we write $\mu = \mu(\Lambda_{1,n,t,b})$. We can write

$$h_\mu \left(R_i \left| \bigcup_{j=1}^{i-1} R_j \right. \right) = \lim_{k \rightarrow \infty} (1/k) S_k$$

where

$$S_k := \sum_{\substack{w \in \mathcal{A}^{[-k,k] \times [1,i-1]}, \\ v \in \mathcal{A}^{[-k,k] \times \{i\}}}} \mu(w \cap v) \log \left(\frac{\mu(w)}{\mu(w \cap v)} \right) \\ = \left(\sum_{w \in \mathcal{A}^{[-k,k] \times [1,i-1]}} \mu(w) \log \mu(w) \right) \tag{2}$$

$$- \left(\sum_{\substack{w \in \mathcal{A}^{[-k,k] \times [1,i-1]}, \\ v \in \mathcal{A}^{[-k,k] \times \{i\}}} \mu(w \cap v) \log \mu(w \cap v) \right). \tag{3}$$

(As in the definition of measure-theoretic entropy, in each sum we omit terms coming from configurations of μ -measure zero.) We also define

$$S_k^* := \sum_{\substack{w \in \mathcal{A}^{[-k,k] \times [1,i-1]}, \\ v \in \mathcal{A}^{[-k,k] \times \{i\}}, \\ L \in \mathcal{A}^{[-k-1] \times [1,i-1]}, \\ R \in \mathcal{A}^{[k+1] \times [1,i-1]}}} \mu(w \cap v \cap L \cap R) \log \left(\frac{\mu(w \cap L \cap R)}{\mu(w \cap v \cap L \cap R)} \right) \\ = \left(\sum_{\substack{w \in \mathcal{A}^{[-k,k] \times [1,i-1]}, \\ L \in \mathcal{A}^{[-k-1] \times [1,i-1]}, \\ R \in \mathcal{A}^{[k+1] \times [1,i-1]}}} \mu(w \cap L \cap R) \log \mu(w \cap L \cap R) \right) \tag{4}$$

$$- \left(\sum_{\substack{w \in \mathcal{A}^{[-k,k] \times [1,i-1]}, \\ v \in \mathcal{A}^{[-k,k] \times \{i\}}, \\ L \in \mathcal{A}^{[-k-1] \times [1,i-1]}, \\ R \in \mathcal{A}^{[k+1] \times [1,i-1]}}} \mu(w \cap v \cap L \cap R) \log \mu(w \cap v \cap L \cap R) \right). \tag{5}$$

We claim that $|S_k - S_k^*| \leq 4(i - 1) \log |\mathcal{A}|$. To see this, we first compare (2) and (4). For any fixed w , consider the term $\mu(w) \log \mu(w)$ from (2). Compare this to the corresponding terms from (4), i.e. $\sum_{L,R} \mu(w \cap L \cap R) \log \mu(w \cap L \cap R)$. We make

the simple observation that for any set of k non-negative reals $\{\alpha_i\}_{i=1}^k$ summing to α , $-\sum_{i=1}^k \alpha_i \log(\alpha_i)$ is at least $-\alpha \log \alpha$, and at most $-\alpha \log(\alpha/k)$ (achieved when all α_i are equal). Therefore,

$$\left| \mu(w) \log \mu(w) - \sum_{L,R} \mu(w \cap L \cap R) \log \mu(w \cap L \cap R) \right| \leq \mu(w) \log(|\mathcal{A}|^{2(i-1)})$$

since the number of different pairs L, R is at most $|\mathcal{A}|^{2(i-1)}$. By summing this over all choices of w in (2) and (4), we see that the difference between (2) and (4) has absolute value at most $\sum_w \mu(w) \log(|\mathcal{A}|^{2(i-1)}) = 2(i-1) \log |\mathcal{A}|$. An analogous argument may be made for (3) and (5), and so $|S_k - S_k^*| \leq 4(i-1) \log |\mathcal{A}|$ for all k .

We can similarly write $h_\mu(R_i | R_{i-1}) = \lim_{k \rightarrow \infty} (1/k) T_k$, where T_k is defined exactly as S_k , but where $w \in \mathcal{A}^{[-k,k] \times [1,i-1]}$ is replaced in the summation by $w' \in \mathcal{A}^{[-k,k] \times \{i\}}$. If we define T_k^* exactly as S_k^* , again using w' instead of w , then a trivially similar proof to the above shows that $|T_k - T_k^*| \leq 4(i-1) \log |\mathcal{A}|$ for all k . If we can now prove that $S_k^* = T_k^*$ for all k , then

$$|S_k - T_k| \leq |S_k - S_k^*| + |T_k - T_k^*| + |S_k^* - T_k^*| \leq 8(i-1) \log |\mathcal{A}|$$

for every k , which clearly shows that

$$h_\mu \left(R_i \left| \bigcup_{j=1}^{i-1} R_j \right. \right) = \lim_{k \rightarrow \infty} (1/k) S_k \quad \text{and} \quad h_\mu(R_i | R_{i-1}) = \lim_{k \rightarrow \infty} (1/k) T_k$$

are equal.

We claim that for any $L \in \mathcal{A}^{[-k-1] \times [1,i-1]}$, $R \in \mathcal{A}^{[k+1] \times [1,i-1]}$, $v \in \mathcal{A}^{[-k,k] \times \{i\}}$, and $w \in \mathcal{A}^{[-k,k] \times [1,i-1]}$, if we define $w' = w|_{[-k,k] \times \{i-1\}}$, then

$$\frac{\mu(w \cap L \cap R)}{\mu(w \cap v \cap L \cap R)} = \frac{\mu(w' \cap L \cap R)}{\mu(w' \cap v \cap L \cap R)}.$$

To see this, we define $w'' = w|_{[-k,k] \times [1,i-2]}$ and note that since μ is an $H_{m,n}$ -MRF,

$$\frac{\mu(w'' \cap w' \cap L \cap R)}{\mu(w' \cap L \cap R)} = \frac{\mu(w'' \cap w' \cap v \cap L \cap R)}{\mu(w' \cap v \cap L \cap R)}.$$

From this, it is clear that

$$\frac{\mu(w' \cap L \cap R)}{\mu(w' \cap v \cap L \cap R)} = \frac{\mu(w'' \cap w' \cap L \cap R)}{\mu(w'' \cap w' \cap v \cap L \cap R)},$$

which is clearly equal to $\mu(w \cap L \cap R) / \mu(w \cap v \cap L \cap R)$. Then, however, for any fixed w' , all terms in S_k^* corresponding to w with the top row w' can be collapsed, which quickly yields $S_k^* = T_k^*$, completing the proof. \square

COROLLARY 3.19. *If Λ is a translation-invariant \mathbb{Z}^2 -specification with $q(\Lambda) < p_c$ (with unique associated MRF $\mu = \mu(\Lambda)$), and t and b are constant sequences such that $-n, n, t, b$ is compatible with Λ for sufficiently large n , and k is any positive integer, then*

$$h_\mu \left(R_0 \left| \bigcup_{j=-k}^{-1} R_j \right. \right) = h_\mu(R_0 | R_{-1}).$$

Proof. Fix any such μ, t, b , and $N > 0$ such that $-n, n, t, b$ is compatible with Λ for $n > N$. Let k be a positive integer. Then by Theorem 3.16, there exist $K, L > 0$ so that

$$\bar{d}(\mu(\Lambda_{-n,n,t,b})|_{\bigcup_{j=-(k-1)}^0 R_j}, \mu(\Lambda_{-(n+1),n+1,t,b})|_{\bigcup_{j=-(k-1)}^0 R_j}) < kKe^{-L(n-k)}$$

for any $n > N$ and $k \leq n$, and so this sequence of marginalizations is \bar{d} Cauchy and approaches a \bar{d} limit. Since the measures $\mu(\Lambda_{-n,n,t,b})$ approach μ weakly as $n \rightarrow \infty$ by Proposition 3.15, it must be the case that the \bar{d} limit of $\mu(\Lambda_{-n,n,t,b})|_{\bigcup_{j=-k}^0 R_j}$ is $\mu|_{\bigcup_{j=-k}^0 R_j}$. Thus, by Theorem 3.18, translation-invariance of μ , and continuity of entropy with respect to the \bar{d} metric, we conclude that for any positive integer k , $h_\mu(R_0| \bigcup_{j=-k}^{-1} R_j) = h_\mu(R_0|R_{-1})$. \square

COROLLARY 3.20. *If Λ is a translation-invariant \mathbb{Z}^2 -specification with $q(\Lambda) < p_c$ (with unique associated MRF $\mu = \mu(\Lambda)$), and t and b are constant sequences such that $-n, n, t, b$ is compatible with Λ for sufficiently large n , then $h(\mu) = h_\mu(R_0|R_{-1})$.*

Proof. For any such μ and positive integer k ,

$$\begin{aligned} h_\mu\left(\bigcup_{j=0}^k R_j\right) &= h_\mu(R_0) + h_\mu(R_1|R_0) + \dots + h_\mu\left(R_k \mid \bigcup_{j=0}^{k-1} R_j\right) \\ &= h_\mu(R_0) + (k-1)h_\mu(R_0|R_{-1}) \end{aligned}$$

by translation-invariance of μ , Proposition 3.17, and Corollary 3.19. Then, however, by dividing by k and letting $k \rightarrow \infty$, we see that $h(\mu) = h_\mu(R_0|R_{-1})$. \square

THEOREM 3.21. *For any valid translation-invariant \mathbb{Z}^2 -specification Λ with $q(\Lambda) < p_c$ (with unique associated MRF $\mu = \mu(\Lambda)$) and any integer N and constant sequences t and b such that $1, n, t, b$ is compatible with Λ when $n > N$, there exist $Q, R > 0$ such that $|h(\mu(\Lambda_{1,n+1,t,b})) - h(\mu(\Lambda_{1,n,t,b})) - h(\mu)| < Qe^{-Rn}$ for any $n > N$.*

Proof. Proposition 3.17 and Theorem 3.18 imply that for $n > N$,

$$h(\mu(\Lambda_{1,n+1,t,b})) = h_{\mu(\Lambda_{1,n+1,t,b})}(R_1) + \sum_{j=1}^n h_{\mu(\Lambda_{1,n+1,t,b})}(R_{j+1}|R_j)$$

and

$$h(\mu(\Lambda_{1,n,t,b})) = h_{\mu(\Lambda_{1,n,t,b})}(R_1) + \sum_{j=1}^{n-1} h_{\mu(\Lambda_{1,n,t,b})}(R_{j+1}|R_j).$$

We can then write $h(\mu(\Lambda_{1,n+1,t,b})) - h(\mu(\Lambda_{1,n,t,b}))$ as

$$h_{\mu(\Lambda_{1,n+1,t,b})}(R_1) - h_{\mu(\Lambda_{1,n,t,b})}(R_1) \tag{6}$$

$$+ \sum_{j=1}^{\lfloor n/2 \rfloor - 1} (h_{\mu(\Lambda_{1,n+1,t,b})}(R_{j+1}|R_j) - h_{\mu(\Lambda_{1,n,t,b})}(R_{j+1}|R_j)) \tag{7}$$

$$+ h_{\mu(\Lambda_{1,n+1,t,b})}(R_{\lfloor n/2 \rfloor + 1} | R_{\lfloor n/2 \rfloor}) \tag{8}$$

$$+ \sum_{j=\lfloor n/2 \rfloor + 1}^n (h_{\mu(\Lambda_{1,n+1,t,b})}(R_{j+1}|R_j) - h_{\mu(\Lambda_{1,n,t,b})}(R_j|R_{j-1})). \tag{9}$$

By Theorem 3.16 and Hölder continuity of entropy with respect to \bar{d} , we see that (6), (7), and (9) are exponentially small in n , i.e. there exist constants Q and R independent of n such that each has absolute value smaller than Qe^{-Rn} . Theorem 3.16 also implies that the sequence $\mu(\Lambda_{1,n+1,t,b})|_{R_{\lfloor n/2 \rfloor} \cup R_{\lfloor n/2 \rfloor + 1}}$ approaches a \bar{d} limit with exponential rate in n , as $n \rightarrow \infty$. Since $\sigma_{(0, -\lfloor n/2 \rfloor - 1)}\mu(\Lambda_{1,n+1,t,b})$ approaches μ weakly by Proposition 3.15, this \bar{d} limit must be $\mu|_{R_{-1} \cup R_0}$. Then, however, again by Hölder continuity of entropy with respect to \bar{d} and translation-invariance of μ , (8) approaches $h_\mu(R_0|R_{-1})$ with exponential rate as $n \rightarrow \infty$, which equals $h(\mu)$ by Corollary 3.20. \square

This result can be generalized to periodic boundary conditions as well. If Λ is a translation-invariant \mathbb{Z}^2 -specification with $q(\Lambda) < p_c$ and t and b are periodic sequences such that $1, n, t, b$ is compatible with Λ , then $\mu(\Lambda_{1,n,t,b})$ exists by Propositions 3.13 and 3.14, and $(\mu(\Lambda_{1,n,t,b}))^{[p]}$ (the p -higher power code of $\mu(\Lambda_{1,n,t,b})$) is translation-invariant.

THEOREM 3.22. *For any valid translation-invariant \mathbb{Z}^2 -specification Λ with $q(\Lambda) < p_c$ (with unique associated MRF $\mu = \mu(\Lambda)$), and any integer N and sequences t and b with period p such that $1, n, t, b$ is compatible with Λ when $n > N$, there exist $Q, R > 0$ such that $|(1/p)(h((\mu(\Lambda_{1,n+1,t,b}))^{[p]}) - h((\mu(\Lambda_{1,n,t,b}))^{[p]}) - h(\mu))| < Qe^{-Rn}$ for all $n > N$.*

Proof. The proof is nearly identical to that of Theorem 3.21, and so we only highlight the slight differences. Theorem 3.16 implies the exponential \bar{d} closeness of relevant marginalizations of $\mu(\Lambda_{1,n,t,b})$ to strips just as before. Then, since passing to $(\mu(\Lambda_{1,n,t,b}))^{[p]}$ multiplies the relevant \bar{d} distances by at most p , we still have the necessary exponential \bar{d} closeness of marginalizations for these recoded strip measures. Since the $(\mu(\Lambda_{1,n,t,b}))^{[p]}$ are translation-invariant, the same proof as in Theorem 3.21 shows that $h((\mu(\Lambda_{1,n+1,t,b}))^{[p]}) - h((\mu(\Lambda_{1,n,t,b}))^{[p]})$ approaches $h(\mu^{[p]}) = ph(\mu)$ exponentially fast. \square

4. Interactions and Gibbs states

In [P], similar techniques were used to show that the topological entropy of the \mathbb{Z}^2 hard square shift \mathcal{H} is exponentially well approximable by differences of consecutive topological entropies of horizontal bi-infinite strips. It turns out that this is a corollary of Theorem 3.21; the unique measure of maximal entropy μ for \mathcal{H} is in fact the unique MRF associated to a translation-invariant \mathbb{Z}^2 -specification Λ satisfying $q(\Lambda) < p_c$; in this case, for any finite set S and $\delta \in \mathcal{A}^{\partial S}$, Λ^δ will be uniform on configurations which are locally admissible in $S \cup \partial S$. If one takes $t = b = 0^\infty$, then $\mu(\Lambda_{1,n,t,b})$ exists for all n , and is the unique measure of maximal entropy for the nearest neighbor \mathbb{Z} -SFT composed of all locally admissible configurations on $H_{1,n}$ in \mathcal{H} .

We will use Theorem 3.22 to generalize the main result of [P] to some topological pressures by using some classical results of Ruelle regarding the relationship between equilibrium states and a class of \mathbb{Z}^2 -MRFs called Gibbs states.

Let X be a non-empty nearest neighbor \mathbb{Z}^d -SFT X with language $\mathcal{L}(X)$. We are mostly interested in the cases $d = 1, 2$. An *interaction* on X is simply a real-valued function Φ on $\mathcal{L}(X)$. The interaction is *finite-range* if there exists M such that Φ vanishes on $\mathcal{L}_S(X)$ for

any S with diameter larger than M . It is *translation-invariant* if it assigns the same values to all translates of a given configuration on a finite set.

For finite S and $x \in \mathcal{L}_S(X)$, define

$$U_S(x) = U_S^\Phi(x) = \sum_{S' \subseteq S} \Phi(x|_{S'}).$$

For $y \in \mathcal{A}^{\mathbb{Z}^d \setminus S}$ such that $xy \in X$, define

$$W_S(x, y) = W_S^\Phi(x, y) = \sum_{T \subseteq \mathbb{Z}^d, |T| < \infty: T \cap S \neq \emptyset, T \cap \mathbb{Z}^d \setminus S \neq \emptyset} \Phi((xy)|_T).$$

A *Gibbs state* for a translation-invariant finite-range interaction Φ on X is a measure μ with support contained in X such that for any $x \in \mathcal{L}_S(X)$ and $y \in \mathcal{A}^{\mathbb{Z}^d \setminus S}$ with $xy \in X$,

$$\mu(x|y) = \frac{\exp(-U_S(x) - W_S(x, y))}{\sum_{\{x' \in \mathcal{L}_S(X): x'y \in X\}} \exp(-U_S(x') - W_S(x', y))} \tag{10}$$

(note that the denominator is simply a normalization factor). The measure itself is given as an integral with (10) as the integrand. This agrees with the classical definition of Gibbs state (see [Rue, Ch. 1, (1.15)]) in the case of a nearest neighbor SFT. Ruelle [Rue, Ch. 1] shows that given any such Φ and X , there is at least one Gibbs state μ (Ruelle’s result is actually much more general).

An interaction is a *nearest neighbor interaction* if it vanishes on all configurations other than those on vertices and edges.

We now define a specification corresponding to (10) for translation-invariant, nearest neighbor interactions. For a finite set $S \subset \mathbb{Z}^d$ and $z \in \mathcal{L}_S(X)$, let

$$\Phi(z, S) = \sum_{u \in S} \Phi(z(u)).$$

For a finite set E of edges in $E(\mathbb{Z}^d)$ such that $\cup_{e \in E} e \subset S$ and $z \in \mathcal{L}_S(X)$, let

$$\Phi(z, E) = \sum_{e \in E} \Phi(z|_e).$$

The *Gibbs \mathbb{Z}^d -specification determined by Φ and X* , denoted by $\Lambda_{\Phi, X}$, is defined as follows. For any finite $S \subset \mathbb{Z}^d$, $x \in \mathcal{A}^S$, and $\delta \in \mathcal{L}_{\partial S}(X)$,

$$\Lambda_{\Phi, X}^\delta(x) = \frac{\exp(-\Phi(x, S) - \Phi(x\delta, E(S, S) \cup E(S, \partial S)))}{\sum_{\{x' \in \mathcal{A}^S: x'\delta \in \mathcal{L}_{S \cup \partial S}(X)\}} \exp(-\Phi(x', S) - \Phi(x'\delta, E(S, S) \cup E(S, \partial S)))} \tag{11}$$

if $x\delta \in \mathcal{L}_{S \cup \partial S}(X)$, and $\Lambda_{\Phi, X}^\delta(x) = 0$ otherwise. For $\delta \in \mathcal{A}^{\partial S} \setminus \mathcal{L}_{\partial S}(X)$, set $\Lambda_{\Phi, X}^\delta = \Lambda_{\Phi, X}^{\delta'}$ for some arbitrary $\delta' \in \mathcal{L}_{\partial S}(X)$ (note that $\mathcal{L}_{\partial S}(X)$ is non-empty since X is non-empty). In this case $q(\Lambda_{\Phi, X})$ is determined only by $\{\Lambda_{\Phi, X}^\delta\}_{\delta \in \mathcal{L}(X)}$.

If $x \in \mathcal{L}_S(X)$ and $y \in \mathcal{A}^{\mathbb{Z}^d \setminus S}$ with $xy \in X$ and $\delta = y|_{\partial S}$, then

$$U_S(x) = \Phi(x, S) + \Phi(x, E(S, S)),$$

$$W_S(x, y) = \Phi(x\delta, E(S, \partial S)),$$

and so for any Gibbs state μ for Φ on X the conditional probabilities (10) reduce to $\mu(x|y) = \Lambda_{\Phi, X}^\delta(x)$. Since μ is supported in X , all other conditional probabilities $\mu(x|y)$

are spurious, and so μ is a \mathbb{Z}^d -MRF associated to $\Lambda_{\Phi, X}$. Since there always exists a Gibbs state for Φ and X (by Ruelle’s result above), $\Lambda_{\Phi, X}$ is valid. We also note that since Φ was assumed to be translation-invariant, $\Lambda_{\Phi, X}$ is translation-invariant as well.

Given a translation-invariant nearest neighbor interaction Φ on a nearest neighbor \mathbb{Z}^2 -SFT X , we will find it useful to represent a corresponding Gibbs state as a Gibbs state for an interaction which is non-zero on a single finite set, namely the set $\Delta = \{(0, 0), (0, 1), (1, 0)\}$. Specifically, define

$$\hat{\Phi}(x) = \Phi(x|_{\{(0,0)\}}) + \Phi(x|_{\{(0,0),(0,1)\}}) + \Phi(x|_{\{(0,0),(1,0)\}}) \tag{12}$$

for $x \in \mathcal{L}_\Delta(X)$.

PROPOSITION 4.1. Any Gibbs state for $\hat{\Phi}$ on X is a Gibbs state for Φ on X .

Proof. For a horizontal edge $e = \{u, u + (1, 0)\}$ let $e^\perp = \{u, u + (0, 1)\}$. For a vertical edge $e = \{u, u + (0, 1)\}$ let $e^\perp = \{u, u + (1, 0)\}$. For a finite set S , let $D(S) = \{\text{edges } e : e \cap S = \emptyset \text{ and } e^\perp \in E(S, \partial S)\}$.

For $x \in \mathcal{A}^S, y \in \mathcal{A}^{\mathbb{Z}^d \setminus S}$, if $xy \in X$, then letting $\delta = y|_{\partial S}$, we have

$$U_S^{\hat{\Phi}}(x) + W_S^{\hat{\Phi}}(x, y) = \Phi(x, S) + \Phi(x\delta, E(S, S) \cup E(S, \partial S)) + \Phi(xy, D(S)).$$

Note that the last term in this expression is the same if we replace x by any $x' \in \mathcal{A}^S$ such that $x'\delta \in \mathcal{L}_{S \cup \partial S}(X)$. It follows that for a Gibbs state for $\hat{\Phi}$ on X , the conditional probabilities (10) reduce to the specification determined by Φ and X . \square

We conclude this section by presenting two interactions on nearest neighbor \mathbb{Z}^2 -SFTs which define historically important Gibbs measures. We will return to these examples later in the paper to demonstrate how later results apply to them.

- (1) The \mathbb{Z}^2 *hard-core model* with activity a is given by X equal to the \mathbb{Z}^2 hard square shift \mathcal{H} and translation-invariant nearest neighbor interaction Φ defined by $\Phi(u) = au(v)$ for u a configuration on a vertex v and $\Phi(u) = 0$ for u a configuration on an edge. In this model, a can be thought of as the ‘weight’ given to the symbol 1.
- (2) The \mathbb{Z}^2 *Ising antiferromagnet* with external magnetic field h and temperature $T = 1/\beta$ is defined by $X = \{\pm 1\}^{\mathbb{Z}^2}$ and translation-invariant nearest neighbor interaction Φ defined by $\Phi(u) = -\beta hu(v)$ for u a configuration on a vertex v and $\Phi(u) = \beta u(v_1)u(v_2)$ for u a configuration on an edge $\{v_1, v_2\}$. In this model, h is an external influence which gives individual sites a preference between 1 and -1 , and β can be thought of as the ‘penalty’ imposed to aligned adjacent sites; when β is very large, adjacent sites are more likely to differ.

5. Strongly irreducible SFTs

We from now on restrict our attention to a specific class of nearest neighbor \mathbb{Z}^2 -SFTs with very strong topological mixing properties.

Definition 5.1. A nearest neighbor \mathbb{Z}^2 -SFT X is strongly irreducible (with filling distance L) if for any finite $S, T \subset \mathbb{Z}^2$ with $d(S, T) > L$, and any $x \in \mathcal{L}_S(X), y \in \mathcal{L}_T(X)$, it is always the case that $xy \in \mathcal{L}_{S \cup T}(X)$.

Our first use of strong irreducibility is to present a sufficient condition for a \mathbb{Z}^2 -MRF to be fully supported within an SFT.

PROPOSITION 5.2. *If X is a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT and Λ is a valid \mathbb{Z}^2 -specification such that $x\delta \in \mathcal{L}(X)$ implies $\Lambda^\delta(x) > 0$, then any \mathbb{Z}^2 -MRF associated to Λ whose support is contained in X is fully supported on X .*

Proof. Denote by L the filling distance of X . Let μ be any \mathbb{Z}^2 -MRF associated to Λ whose support is contained in X . Fix any $w \in \mathcal{L}_{[-n,n]^2}(X)$ and any $\delta \in \mathcal{A}^{\partial([-n-L,n+L]^2)}$ such that δ has positive μ -measure. Since the support of μ is contained in X , $\delta \in \mathcal{L}(X)$. Therefore, by strong irreducibility of X , there exists $x \in \mathcal{L}_{[-n-L,n+L]^2}(X)$ with $x|_{[-n,n]^2} = w$ and $x\delta \in \mathcal{L}(X)$. By the assumption on Λ , $\Lambda^\delta(x) > 0$, and since μ is associated to Λ , this means that $\mu^\delta(x) > 0$. Then, however, since $\mu(\delta)$ is positive, $\mu(x)$ is as well. Since w is a subconfiguration of x , $\mu(w) > 0$. \square

From this, we can conclude a fact about certain valid \mathbb{Z}^2 -specifications supported on strongly irreducible nearest neighbor \mathbb{Z}^2 -SFTs that will be useful later.

PROPOSITION 5.3. *Let X be a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT and Λ be a valid \mathbb{Z}^2 -specification such that $x\delta \in \mathcal{L}(X)$ implies $\Lambda^\delta(x) > 0$. Assume that there exists a \mathbb{Z}^2 -MRF associated to Λ whose support is contained in X and that $q(\Lambda) < 1$. Then for any rectangle R and $\delta, \delta' \in \mathcal{L}_{\partial R}(X)$, there exists $u \in \mathcal{A}^R$ such that $\delta u, \delta' u \in \mathcal{L}(X)$.*

Proof. Consider such $X, \Lambda, R, \delta,$ and δ' , and let μ be a \mathbb{Z}^2 -MRF associated to Λ with support contained in X . By Proposition 5.2, μ is fully supported on X .

Without loss of generality, we assume $R = [0, m] \times [0, n]$ for some non-negative m, n . We begin by proving the proposition for $m = n = 0$, i.e. $R = \{(0, 0)\}$. Consider any $\zeta, \zeta' \in \mathcal{L}_{N_{(0,0)}}(X)$. Then by full support of μ on X , $\mu(\zeta), \mu(\zeta') > 0$. Since $q(\Lambda) < 1$, there exists $a \in \mathcal{A}^{\{(0,0)\}}$ such that $\Lambda^\zeta(a), \Lambda^{\zeta'}(a) > 0$, and since μ is associated to Λ , $\mu^\zeta(a), \mu^{\zeta'}(a) > 0$. Therefore, $\mu(\zeta a), \mu(\zeta' a) > 0$, and since the support of μ is contained in X , $\zeta a, \zeta' a \in \mathcal{L}(X)$.

We will use this fact to deal with general R . Define by r_1, r_2, \dots, r_{mn} the elements of R , listed in lexicographic order (i.e. the bottom row from left to right, then the next row from left to right, etc.) For each $k \in [0, mn]$, define $R_k = \{r_i\}_{i=1}^k$. We define u letter by letter, on the sites r_i in order. Suppose that for some $k \in [0, mn]$, we have defined $u_k \in \mathcal{A}^{R_k}$ such that $\delta u_k, \delta' u_k \in \mathcal{L}(X)$. We must define $u_{k+1} \in \mathcal{A}^{R_{k+1}}$ such that $u_{k+1}|_{R_k} = u_k$ and $\delta u_{k+1}, \delta' u_{k+1} \in \mathcal{L}(X)$.

Define $C = N_{r_{k+1}} \setminus (R_k \cup \partial R)$, the set of neighbors of r_{k+1} which do not have letters assigned to them in δ, δ' , or u_k . Clearly, since δu_k and $\delta' u_k$ are globally admissible in X , there exist $\eta, \eta' \in \mathcal{L}_C(X)$ such that $\delta \eta u_k, \delta' \eta' u_k \in \mathcal{L}(X)$. Define $\zeta = (\delta \eta u_k)|_{N_{r_{k+1}}}$ and $\zeta' = (\delta' \eta' u_k)|_{N_{r_{k+1}}}$; clearly $\zeta, \zeta' \in \mathcal{L}(X)$.

Then by the proof of the proposition for R consisting of a single site, there exists $a \in \mathcal{L}_{\{r_{k+1}\}}(X)$ such that $\zeta a, \zeta' a \in \mathcal{L}(X)$. Then from the facts that X is a nearest neighbor SFT, that $\zeta, \zeta' \in \mathcal{A}^{N_{r_{k+1}}}$, and that $\delta \eta u_k$ and $\delta' \eta' u_k$ are globally admissible in X , we can conclude that $\delta \eta u_k a$ and $\delta' \eta' u_k a$ are globally admissible in X as well. Then, however, trivially $\delta u_k a, \delta' u_k a \in \mathcal{L}(X)$, and so we can take $u_{k+1} = u_k a$.

This inductive process eventually yields $u_{mn} \in \mathcal{A}^R$ for which $\delta u_{mn}, \delta' u_{mn} \in \mathcal{L}(X)$, and so by taking $u = u_{mn}$ we are done. \square

Our main application of strong irreducibility is, for any Gibbs specification $\Lambda_{\Phi, X}$, to guarantee the existence of periodic rows t, b such that $(\Lambda_{\Phi, X})_{m, n, t, b}$ is valid for $n - m$ large (recall that this is an induced specification on the strip subgraph $H_{m, n}$).

We first recall the result of Ward [War] that any strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT X has a globally admissible periodic row (i.e., a periodic configuration on $\mathbb{Z} \times \{0\}$ which extends to an element of X); in fact, every such SFT has a doubly periodic element, though we will not need this fact here.

For integers $m < n$ and globally admissible periodic rows t, b , let $X_{m, n, t, b}$ be the set of all configurations $x \in \mathcal{A}^{\mathbb{Z} \times [m, n]}$ such that txb is locally admissible. Note that all elements of $X_{m, n, t, b}$ are in fact globally admissible.

PROPOSITION 5.4. *Let X be a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT and Φ be a translation-invariant nearest neighbor interaction on X . Let t and b be periodic rows which are globally admissible in X (which always exist by [War]). If $n - m$ exceeds the filling distance of X , then m, n, t, b is compatible with $\Lambda_{\Phi, X}$ (and therefore by Proposition 3.13 $(\Lambda_{\Phi, X})_{m, n, t, b}$ is valid).*

Moreover, there is an MRF associated to $(\Lambda_{\Phi, X})_{m, n, t, b}$ supported in $X_{m, n, t, b}$.

Proof. Let t and b be such rows. Since $\Lambda_{\Phi, X}$ is Gibbs, it is valid, and so has an associated MRF μ supported in X . Let m, n be chosen so that $n - m$ exceeds the filling distance of X . Then for all k , there exists $w_k \in \mathcal{L}_{[-k, k] \times [m-1, n+1]}(X)$ whose top row is $t|_{[-k, k]}$ and bottom row is $b|_{[-k, k]}$. Proposition 5.2 applies to μ , and so $\mu(w_k) > 0$. Thus, m, n, t, b is compatible with $\Lambda_{\Phi, X}$ (by taking $\delta_k = w_k|_{\partial([-k+1, k-1] \times [m, n])}$ in the definition of compatibility).

The measure obtained as in Proposition 3.13 from the δ_k is clearly supported in $X_{m, n, t, b}$. \square

For any translation-invariant nearest neighbor interaction Φ , strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT X with filling distance L , globally admissible periodic t and b , and m, n with $n - m > L$, for notational convenience we denote by $\Lambda_{\Phi, X, m, n, t, b}$ the specification $(\Lambda_{\Phi, X})_{m, n, t, b}$ which by Proposition 5.4 is valid. If additionally $q(\Lambda) < 1$, then by Proposition 3.14 there is a unique $H_{m, n}$ -MRF $\mu(\Lambda_{\Phi, X, m, n, t, b})$ associated to $\Lambda_{\Phi, X, m, n, t, b}$, which we call the *induced Gibbs state* for Φ and X , and which, again for convenience, we denote by $\mu_{\Phi, X, m, n, t, b}$. By Proposition 5.4 $\mu_{\Phi, X, m, n, t, b}$ is supported in $X_{m, n, t, b}$.

6. Induced Gibbs states as one-dimensional Markov chains

Our eventual goal is to use the results of §§3, 4, and 5 to give an algorithm for approximating certain topological pressures by means of induced Gibbs states on strips. For this procedure to be useful though, it must be the case that these induced Gibbs states are tractable measures to deal with. Under some elementary assumptions, this does turn out to be the case; in fact the induced Gibbs states turn out to be Markov chains.

Let X be a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT with filling distance L , Φ a translation-invariant nearest neighbor interaction on X such that $q(\Lambda_{\Phi, X}) < 1$, and let μ

be any Gibbs state for Φ on X . Let t and b be globally admissible periodic rows for X , which for now we assume to be constant (though such rows may not exist in general): $t = t_0^\infty, b = b_0^\infty$. Let m, n be integers such that $n - m > L$.

Let $\mathcal{C}_{m,n,t,b}$ denote the set of all ‘locally admissible columns compatible with t and b ’ on $H_{m,n}$, i.e., all $x_m \dots x_n$ such that $x_i x_{i+1} \in \mathcal{E}_2$ for $i = m, \dots, n - 1$, and $b_0 x_m, x_n t_0 \in \mathcal{E}_2$. Let $\mathcal{E}_{m,n,t,b}$ denote the set of all locally admissible $(n - m + 1) \times 2$ rectangles, i.e., ordered pairs of columns $(x_m \dots x_n, y_m \dots y_n) \in (\mathcal{C}_{m,n,t,b})^2$ such that $x_i y_i \in \mathcal{E}_1$ for each $i = m, \dots, n$.

Observe that $X_{m,n,t,b}$ (defined near the end of §5) is the nearest neighbor \mathbb{Z} -SFT on the alphabet $\mathcal{C}_{m,n,t,b}$ defined by $\mathcal{E} = \mathcal{E}_{m,n,t,b}$. Here, we are identifying a configuration on a finite interval of \mathbb{Z} over the alphabet $\mathcal{C}_{m,n,t,b}$ with the corresponding configuration on a finite rectangle in $\mathbb{Z} \times [m, n]$ over the alphabet \mathcal{A} . Recall that since t and b are globally admissible and X is a nearest neighbor \mathbb{Z}^2 -SFT, we have $\mathcal{L}(X_{m,n,t,b}) \subset \mathcal{L}(X)$.

Denote by $\overline{\Lambda}_{\Phi, X, m, n, t, b}$ the restriction of the specification $\Lambda_{\Phi, X, m, n, t, b}$ to configurations δ of the form $\partial(F \times [m, n], H_{m,n})$ for finite sets F . We think of $\overline{\Lambda}_{\Phi, X, m, n, t, b}$ as a \mathbb{Z} -specification over the alphabet $\mathcal{A}^{[m,n]}$.

COROLLARY 6.1. *If X is a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT, Φ is a translation-invariant nearest neighbor interaction for which $q(\Lambda_{\Phi, X}) < 1$, t, b are globally admissible constant sequences, and $n - m$ exceeds the filling distance of X , then $q(\overline{\Lambda}_{\Phi, X, m, n, t, b}) < 1$.*

Proof. To show that $q(\overline{\Lambda}_{\Phi, X, m, n, t, b}) < 1$, it suffices to show that for any $c, c' \in \mathcal{L}_{\{-1\}}(X_{m,n,t,b})$ and $d, d' \in \mathcal{L}_{\{1\}}(X_{m,n,t,b})$ such that $cd, c'd' \in \mathcal{L}(X_{m,n,t,b})$, there exists $e \in \mathcal{L}_{\{0\}}(X_{m,n,t,b})$ such that $ced, c'ed' \in \mathcal{L}(X_{m,n,t,b})$. (This is sufficient since $\Lambda_{\Phi, X}$, being a Gibbs specification, has the property that $\Lambda_{\Phi, X}^\delta(u) > 0$ if $u\delta \in \mathcal{L}(X)$.)

However, note that this is equivalent to showing that for any $\delta, \delta' \in \mathcal{L}_{\partial(\{0\} \times [m,n])}(X)$ with t_0 at the top and b_0 at the bottom, there exists $u \in \mathcal{A}^{\{0\} \times [m,n]}(X)$ such that $\delta u, \delta' u \in \mathcal{L}(X)$. This is a straightforward consequence of Proposition 5.3, which can be applied since $\Lambda_{\Phi, X}$ is the Gibbs specification determined by Φ and X and $q(\Lambda_{\Phi, X}) < 1$. \square

COROLLARY 6.2. *If X is a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT, Φ is a translation-invariant nearest neighbor interaction for which $q(\Lambda_{\Phi, X}) < 1$, t, b are globally admissible constant sequences, and $n - m$ exceeds the filling distance of X , then $X_{m,n,t,b}$ is a mixing \mathbb{Z} -SFT.*

Proof. By Corollary 6.1, $q(\overline{\Lambda}_{\Phi, X, m, n, t, b}) < 1$. We prove that $X_{m,n,t,b}$ is mixing by using some well-known facts about the structure of nearest neighbor \mathbb{Z} -SFTs. In particular, we show that $X_{m,n,t,b}$ is irreducible and aperiodic; for more details on these properties see [LM].

We first show that $X_{m,n,t,b}$ is irreducible. Assume for a contradiction that it is not. Then there exist at least two non-trivial irreducible components $C, D \subseteq \mathcal{C}_{m,n,t,b}$. Then, however, we clearly have a contradiction to $q(\overline{\Lambda}_{\Phi, X, m, n, t, b}) < 1$; a boundary configuration consisting of two letters from C can only be filled in a globally admissible way with a letter from C , and the same is true for D . Since C and D were non-trivial components, there exist such boundary configurations δ, δ' which are globally admissible, and,

since $C \cap D = \emptyset$ and $\Lambda_{\Phi, X}$ is supported on X , $d(\overline{\Lambda}_{\Phi, X, m, n, t, b}^\delta, \overline{\Lambda}_{\Phi, X, m, n, t, b}^{\delta'}) = 1$. Therefore, our original assumption was wrong and $X_{m, n, t, b}$ is irreducible.

It remains to show that $X_{m, n, t, b}$ is aperiodic, which is done in the same way. Suppose for a contradiction that $X_{m, n, t, b}$ can be partitioned into period classes $P_1, \dots, P_k, k > 1$. Then if we take δ to be a globally admissible boundary configuration in $X_{m, n, t, b}$ consisting of a letter from P_k on the left and a letter from P_2 on the right, and δ' to be a globally admissible boundary configuration in $X_{m, n, t, b}$ consisting of a letter from P_1 on the left and a letter from $P_{3 \pmod k}$ on the right, then δ and δ' can only be filled with letters from P_1 and P_2 respectively. Again, since $P_1 \cap P_2 = \emptyset$, this contradicts $q(\overline{\Lambda}_{\Phi, X, m, n, t, b}) < 1$, so $X_{m, n, t, b}$ is aperiodic and irreducible, therefore mixing. \square

The induced Gibbs state $\mu_{\Phi, X, m, n, t, b}$ can be viewed as a measure on $(C_{m, n, t, b})^{\mathbb{Z}}$ supported in $X_{m, n, t, b}$, and when viewed in this way, it is a \mathbb{Z} -MRF associated to $\overline{\Lambda}_{\Phi, X, m, n, t, b}$. We will show that when viewed in this way $\mu_{\Phi, X, m, n, t, b}$ is a translation-invariant irreducible first-order Markov chain, with a transition probability matrix defined explicitly in terms of Φ .

We first describe $\overline{\Lambda}_{\Phi, X, m, n, t, b}$ explicitly in terms of Φ . For a configuration z on a finite set $U \subset \mathbb{Z} \times \{n\}$ let

$$\Phi_+(z, U) = \sum_{u \in U} \Phi \begin{pmatrix} t_0 \\ z_u \end{pmatrix}.$$

For a configuration z on a finite set $U \subset \mathbb{Z} \times \{m\}$ let

$$\Phi_-(z, U) = \sum_{u \in U} \Phi \begin{pmatrix} z_u \\ b_0 \end{pmatrix}.$$

Let $R = [-k, k] \times [m, n]$, $x \in \mathcal{A}^R$, $\delta \in \mathcal{A}^{\partial(R, H_{m, n})}$ such that $x\delta \in \mathcal{L}(X_{m, n, t, b})$. From (11) and (1), we have

$$\overline{\Lambda}_{\Phi, X, m, n, t, b}^\delta(x) = \frac{\exp(-A(x, \delta))}{\sum_{\{w \in \mathcal{A}^R : w\delta \in \mathcal{L}(X_{m, n, t, b})\}} \exp(-A(w, \delta))} \tag{13}$$

where

$$A(z, \delta) = \Phi(z, R) + \Phi(z\delta, E(R, R) \cup E(R, \partial(R, H_{m, n}))) + \Phi_+(z, R \cap (\mathbb{Z} \times \{n\})) + \Phi_-(z, R \cap (\mathbb{Z} \times \{m\})). \tag{14}$$

We claim that $\overline{\Lambda}_{\Phi, X, m, n, t, b}$ can be expressed as a \mathbb{Z} -specification determined by a nearest neighbor interaction $\overline{\Phi}_{m, n, t, b}$ on $X_{m, n, t, b}$. We define $\overline{\Phi}_{m, n, t, b}$ to vanish on all finite configurations other than those on edges in $\mathcal{E}_{m, n, t, b}$, and on such edges, it is defined by

$$\begin{aligned} \overline{\Phi}_{m, n, t, b}(x_m \dots x_n, y_m \dots y_n) &= \left(\left(\sum_{i=m}^n \Phi(x_i) \right) + \Phi \begin{pmatrix} x_m \\ b_0 \end{pmatrix} + \left(\sum_{i=m}^{n-1} \Phi \begin{pmatrix} x_{i+1} \\ x_i \end{pmatrix} \right) \right) \\ &\quad + \Phi \begin{pmatrix} t_0 \\ x_n \end{pmatrix} + \left(\sum_{i=m}^n \Phi(x_i, y_i) \right). \end{aligned} \tag{15}$$

Let $R = [-k, k] \times [m, n]$, $x \in \mathcal{A}^R$, $\delta \in \mathcal{A}^{\partial(R, H_{m, n})}$ such that $x\delta \in \mathcal{L}(X_{m, n, t, b})$. Observe that

$$\Lambda_{\overline{\Phi}_{m, n, t, b}, X_{m, n, t, b}}^\delta(x) = \frac{\exp(-B(x, \delta))}{\sum_{\{w \in \mathcal{A}^R : w\delta \in \mathcal{L}(X_{m, n, t, b})\}} \exp(-B(w, \delta))} \tag{16}$$

where

$$\begin{aligned}
 B(z, \delta) &= \sum_{j=-k}^{k-1} (\overline{\Phi}_{m,n,t,b}(z|_{\{j\} \times [m,n]}, z|_{\{j+1\} \times [m,n]}) \\
 &\quad + \overline{\Phi}_{m,n,t,b}(z|_{\{k\} \times [m,n]}, \delta|_{\{k+1\} \times [m,n]}) + \overline{\Phi}_{m,n,t,b}(\delta|_{\{-k-1\} \times [m,n]}, z|_{\{-k\} \times [m,n]}) \\
 &= A(z, \delta) + C(\delta),
 \end{aligned}
 \tag{17}$$

where

$$\begin{aligned}
 C(\delta) &= \left(\sum_{i=m}^n \Phi(\delta(-k-1, i)) \right) + \Phi \left(\begin{matrix} \delta(-k-1, m) \\ b_0 \end{matrix} \right) \\
 &\quad + \left(\sum_{i=m}^{n-1} \Phi \left(\begin{matrix} \delta(-k-1, i+1) \\ \delta(-k-1, i) \end{matrix} \right) \right) + \Phi \left(\begin{matrix} t_0 \\ \delta(-k-1, n) \end{matrix} \right).
 \end{aligned}$$

Comparing (13), (14), (16) and (17), we see that

$$\Lambda_{\overline{\Phi}_{m,n,t,b}, X_{m,n,t,b}}^\delta = \overline{\Lambda}_{\Phi, X, m, n, t, b}^\delta$$

for all $\delta \in \mathcal{L}_{\partial(R, H_{m,n})}(X_{m,n,t,b})$. Since $\mu_{\Phi, X, m, n, t, b}$ is supported in $X_{m,n,t,b}$, it follows that $\mu_{\Phi, X, m, n, t, b}$, when viewed as a \mathbb{Z} -MRF, is a Gibbs state for the interaction $\overline{\Phi}_{m,n,t,b}$ on $X_{m,n,t,b}$.

Let $A_{\Phi, X, m, n, t, b}$ be the square matrix indexed by $\mathcal{C}_{m,n,t,b}$ and defined by

$$(A_{\Phi, X, m, n, t, b})_{c,d} = \begin{cases} e^{-\overline{\Phi}_{m,n,t,b}(c,d)} & \text{if } (c, d) \in \mathcal{E}_{m,n,t,b}, \\ 0 & \text{otherwise.} \end{cases}
 \tag{18}$$

We will frequently suppress the dependence of A on Φ, X, m, n, t, b when it causes no confusion.

By Corollary 6.2, we may assume (by deleting elements of $\mathcal{C}_{m,n,t,b}$ which do not actually appear in $X_{m,n,t,b}$) that A is a primitive matrix. Let $\lambda(A)$ represent the Perron (i.e., largest) eigenvalue of A . By the Perron–Frobenius theory, there are unique (up to scalar multiples) right and left (positive) eigenvectors $u = u_{\Phi, X, m, n, t, b}$ and $v = v_{\Phi, X, m, n, t, b}$ corresponding to $\lambda(A)$.

Let $\Pi = \Pi_{\Phi, X, m, n, t, b}$ (again, dependence will frequently be suppressed) be the (primitive) probability transition matrix indexed by $\mathcal{C}_{m,n,t,b}$ defined by

$$\Pi_{c,d} = \frac{A_{c,d}v(d)}{\lambda(A)v(c)}.
 \tag{19}$$

PROPOSITION 6.3. *Let Φ be a translation-invariant nearest neighbor interaction on a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT X . Let t and b be globally admissible constant rows and assume that $n - m$ exceeds the filling distance of X . Assume that $q(\Lambda_{\Phi, X}) < 1$. Then the induced Gibbs state $\mu_{\Phi, X, m, n, t, b}$ for Φ and X is a one-dimensional translation-invariant mixing first-order Markov chain with probability transition matrix $\Pi_{\Phi, X, m, n, t, b}$.*

Proof. This is a special case of a much more general result [Ge, Theorem 10.25]. However, the proof in our case is much simpler, as follows.

Write $\mu = \mu_{\Phi, X, m, n, t, b}$, $u = u_{\Phi, X, m, n, t, b}$, $v = v_{\Phi, X, m, n, t, b}$ and $\lambda = \lambda(A_{\Phi, X, m, n, t, b})$. We assume that u and v are normalized so that $u \cdot v = 1$. Fix any $k > 0$.

Since μ is a Gibbs state for the interaction $\bar{\Phi}_{m, n, t, b}$ on $X_{m, n, t, b}$, for any positive integers k, ℓ and any $x_{-k}, \dots, x_{-1}, x_0 \in \mathcal{C}_{m, n, t, b}$,

$$\begin{aligned} &\mu(x_0|x_{-1}, \dots, x_{-k}) \\ &= \sum_{x_\ell} \mu(x_\ell|x_{-1}, \dots, x_{-k})\mu(x_0|x_\ell, x_{-1}, \dots, x_{-k}) \\ &= \sum_{x_\ell} \mu(x_\ell|x_{-1}, \dots, x_{-k}) \sum_{x_1, \dots, x_{\ell-1}} \mu(x_0, x_1, \dots, x_{\ell-1}|x_\ell, x_{-1}, \dots, x_{-k}) \\ &= \sum_{x_\ell} \mu(x_\ell|x_{-1}, \dots, x_{-k}) \sum_{x_1, \dots, x_{\ell-1}} \mu(x_0, x_1, \dots, x_{\ell-1}|x_\ell, x_{-1}) \\ &= \sum_{x_\ell} \mu(x_\ell|x_{-1}, \dots, x_{-k}) \sum_{x_1, \dots, x_{\ell-1}} \frac{A_{x_{-1}x_0}A_{x_0x_1} \cdots A_{x_{\ell-1}x_\ell}}{\sum_{x'_0, x'_1, \dots, x'_{\ell-1}} A_{x_{-1}x'_0}A_{x'_0x'_1} \cdots A_{x'_{\ell-1}x_\ell}} \\ &= \sum_{x_\ell} \mu(x_\ell|x_{-1}, \dots, x_{-k}) \frac{A_{x_{-1}x_0}(A^\ell)_{x_0x_\ell}}{(A^{\ell+1})_{x_{-1}x_\ell}}. \end{aligned}$$

Since A is primitive, by [LM, Theorem 4.5.12] we have that

$$\lim_{\ell \rightarrow \infty} \frac{(A^\ell)_{c,d}}{\lambda^\ell} = v_c u_d.$$

Thus, given $\epsilon > 0$, for sufficiently large ℓ , $\mu(x_0|x_{-1}, \dots, x_{-k})$ is within ϵ of

$$\sum_{x_\ell} \mu(x_\ell|x_{-1}, \dots, x_{-k}) \frac{A_{x_{-1}x_0} v_{x_0} u_{x_\ell}}{v_{x_{-1}} u_{x_\ell} \lambda} = \frac{A_{x_{-1}x_0} v_{x_0}}{v_{x_{-1}} \lambda}.$$

Thus,

$$\mu(x_0|x_{-1}, \dots, x_{-k}) = \frac{A_{x_{-1}x_0} v_{x_0}}{v_{x_{-1}} \lambda} = \Pi_{x_{-1}, x_0}.$$

In particular, μ is a translation-invariant mixing first-order Markov chain. □

7. Pressure and equilibrium states

We now turn to our main application of Theorem 3.22: the approximation of certain topological pressures on strongly irreducible nearest neighbor \mathbb{Z}^2 -SFTs.

We recall that for any nearest neighbor \mathbb{Z}^d -SFT X and $f \in C(X)$, an equilibrium state is a translation-invariant measure μ on X for which $h(\mu) + \int f d\mu$ is maximized, and that this maximum $P(f)$ is called the topological pressure of f on X . Our main tool is [Rue, Theorem 4.2], which proves that any equilibrium state is a Gibbs state.

Let X be a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT X and Φ be a translation-invariant nearest neighbor interaction such that $q(\Lambda_{\Phi, X}) < p_c$. Define $\hat{\Phi}$ as in (12). Let $f_\Phi : X \rightarrow R$ be defined by $f_\Phi(x) = -\hat{\Phi}(x|_\Delta)$. In this case, [Rue, Theorem 4.2] shows that any equilibrium state for f_Φ is a Gibbs state for $\hat{\Phi}$. (In fact, this is the reason for defining $\hat{\Phi}$: technically [Rue, Theorem 4.2] applies only to interactions supported on configurations on a single shape.) According to our Proposition 4.1, any equilibrium state for f_Φ is a Gibbs state for Φ as well.

Since X is strongly irreducible, there exist t, b globally admissible periodic rows in X . We for now assume that t, b are constant, and deal with the general periodic case later. For any n which exceeds the filling distance of X , let $\lambda_n = \lambda(A_{\Phi, X, 1, n, t, b})$.

THEOREM 7.1. *Let X be a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT and Φ be a translation-invariant nearest neighbor interaction on X such that $q(\Lambda_{\Phi, X}) < p_c$. Let t and b be globally admissible constant rows. Then there exist constants $Q, R > 0$ such that, for sufficiently large n ,*

$$|\log \lambda_{n+1} - \log \lambda_n - P(f_\Phi)| < Qe^{-Rn}.$$

Proof. Let μ be an equilibrium state for f_Φ , i.e., $P(f_\Phi) = h(\mu) + \int f_\Phi d\mu$. By the discussion above, μ is a Gibbs state for Φ on X and, by Theorem 3.9, is the unique \mathbb{Z}^2 -MRF associated to the \mathbb{Z}^2 -specification $\Lambda = \Lambda_{\Phi, X}$.

Recalling the definition of the interaction $\bar{\Phi}_{1, n, t, b}$ (15), we see that $\bar{\Phi}_{1, n, t, b}$ defines a function on the \mathbb{Z} -SFT $X_{1, n, t, b}$ by $z \mapsto \bar{\Phi}_{1, n, t, b}(z_0, z_1)$. With a slight abuse of notation, we let $\bar{\Phi}_{1, n, t, b}$ denote this function.

Combining Proposition 6.3 with the well-known characterization of unique equilibrium states of locally constant functions as Markov chains (see [Kr, p. 99], [BI]), we see that for any n larger than the filling distance of X , $\mu_{\Phi, X, 1, n, t, b}$ is the unique equilibrium state for $\bar{\Phi}_{1, n, t, b}$ on the \mathbb{Z} -SFT $X_{1, n, t, b}$, and

$$\log \lambda_n = P(\bar{\Phi}_{1, n, t, b}) = h(\mu_{\Phi, X, 1, n, t, b}) + \int \bar{\Phi}_{1, n, t, b} d\mu_{\Phi, X, 1, n, t, b}.$$

By Theorem 3.21 and Proposition 5.4, there exist constants \bar{Q}, \bar{R} such that, for sufficiently large n ,

$$|h(\mu_{\Phi, X, 1, n+1, t, b}) - h(\mu_{\Phi, X, 1, n, t, b}) - h(\mu)| < \bar{Q}e^{-\bar{R}n}.$$

It remains to show that

$$\int \bar{\Phi}_{1, n+1, t, b} d\mu_{\Phi, X, 1, n+1, t, b} - \int \bar{\Phi}_{1, n, t, b} d\mu_{\Phi, X, 1, n, t, b}$$

converges exponentially fast to $\int f_\Phi d\mu$. From the definition of $\bar{\Phi}_{1, n, t, b}$, it follows that

$$\int \bar{\Phi}_{1, n+1, t, b} d\mu_{\Phi, X, 1, n+1, t, b} - \int \bar{\Phi}_{1, n, t, b} d\mu_{\Phi, X, 1, n, t, b}$$

can be decomposed into a sum of the form

$$\begin{aligned} & \sum_x \left(\int F(x) d\mu_{\Phi, X, 1, n+1, t, b} - \int F(x) d\mu_{\Phi, X, 1, n, t, b} \right) \\ & + \sum_{x'} \left(\int F(\sigma_{(0,1)}x') d\mu_{\Phi, X, 1, n+1, t, b} - \int F(x') d\mu_{\Phi, X, 1, n, t, b} \right) \\ & + \int (f_\Phi \circ \sigma_{(0, -\lfloor n/2 \rfloor)}) d\mu_{\Phi, X, 1, n+1, t, b}, \end{aligned} \tag{20}$$

where each x in the first sum is a configuration with shape a vertex or edge contained in $\{0, 1\} \times [1, \lfloor n/2 \rfloor]$, each x' in the second sum is a configuration with shape a vertex or

edge contained in $\{0, 1\} \times [\lfloor n/2 \rfloor, n]$, and $F(x)$ represents any one of the functions $\Phi(x)$, $\Phi_{(b_0)}^x$, or $\Phi_x^{(t_0)}$, as appropriate. (Clearly, in the latter two cases, x must be a configuration on a single site contained in the bottom or top row respectively.)

Since distribution distance is dominated by \bar{d} distance, by Theorem 3.16 there exist $K, L > 0$ such that, for any configuration x with shape a vertex or edge contained in $\{0, 1\} \times [1, \lfloor n/2 \rfloor]$,

$$|\mu_{\Phi, X, 1, n+1, t, b}(x) - \mu_{\Phi, X, 1, n, t, b}(x)| \leq K e^{-Ln/2}. \tag{21}$$

Similarly, for any configuration x with shape a vertex or edge contained in $\{0, 1\} \times [\lfloor n/2 \rfloor, n]$,

$$|\mu_{\Phi, X, 1, n+1, t, b}(\sigma_{(0,1)}x) - \mu_{\Phi, X, 1, n, t, b}(x)| \leq K e^{-Ln/2}. \tag{22}$$

By (21) and (22), if we take $Q' = K \max_x |\Phi(x)|$, then each of the first two sums in (20) is less than $5nQ'e^{-Ln/2}$.

The third term of (20) converges to $\int f_\Phi d\mu$ by Proposition 3.15, and is exponentially Cauchy by (21) and (22). Therefore, there exists Q'' such that for sufficiently large n ,

$$\left| \int (f_\Phi \circ \sigma_{(0, -\lfloor n/2 \rfloor)}) d\mu_{\Phi, X, 1, n+1, t, b} - \int f_\Phi d\mu \right| \leq Q'' e^{-Ln/2}.$$

Therefore, (20) is exponentially close to $\int f_\Phi d\mu$. □

We now consider the general case where t and b are globally admissible periodic (but not necessarily constant) rows with common period p and generalize Theorem 7.1 to this case. Since such t and b always exist, this will yield a way to efficiently approximate $P(f_\Phi)$ for any translation-invariant nearest neighbor interaction Φ on a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT X for which $q(\Lambda_{\Phi, X}) < p_c$.

Let $X^{[p]}$ denote $X^{[0, p-1] \times \{0\}}$, the $([0, p-1] \times \{0\})$ -higher power code of X , which is a nearest neighbor \mathbb{Z}^2 -SFT over the alphabet $\mathcal{A}^{[p]} := \mathcal{L}_{[0, p-1] \times \{0\}}(X)$. Define a new translation-invariant nearest neighbor interaction $\Phi^{[p]}$ on $X^{[p]}$ as follows.

- On vertices,

$$\Phi^{[p]}([x_0 \dots, x_{p-1}]) = \sum_{i=0}^{p-1} \Phi(x_i).$$

- On horizontal edges,

$$\Phi^{[p]}([x_0 \dots, x_{p-1}][y_0 \dots, y_{p-1}]) = \Phi(x_0x_1) + \dots + \Phi(x_{p-1}y_0).$$

- On vertical edges,

$$\Phi^{[p]} \begin{pmatrix} [x_0 \dots, x_{p-1}] \\ [y_0 \dots, y_{p-1}] \end{pmatrix} = \Phi \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \dots + \Phi \begin{pmatrix} x_{p-1} \\ y_{p-1} \end{pmatrix}.$$

Let $t^{[p]}, b^{[p]}$ be the constant rows $(t|_{[0, p-1]})^{\mathbb{Z}}, (b|_{[0, p-1]})^{\mathbb{Z}} \in (\mathcal{A}^{[p]})^{\mathbb{Z}}$. Then $t^{[p]}, b^{[p]}$ are globally admissible in $X^{[p]}$.

If $q(\Lambda_{\Phi^{[p]}, X^{[p]}}) < p_c$, we can simply apply Theorem 7.1 to achieve exponentially converging approximations to $P(f_{\Phi^{[p]}})$, and it is fairly easy to see that $P(f_{\Phi^{[p]}}) = pP(f_\Phi)$, so we would be done. However, by examining the definitions, we see that it

could be the case that $q(\Lambda_{\Phi^{[p]}, X^{[p]}}) > q(\Lambda_{\Phi, X})$, and so we must take a more circuitous route.

As earlier, we can define $\overline{\Phi^{[p]}}_{m,n,t^{[p]},b^{[p]}}$ on $(X^{[p]})_{m,n,t^{[p]},b^{[p]}}$ and the corresponding matrix $A_{\Phi^{[p]}, X^{[p]}, m,n,t^{[p]},b^{[p]}}$. For fixed X, Φ, t, b, p and any n greater than the filling distance of $X^{[p]}$, we make the notation $A_n^{[p]} := A_{\Phi^{[p]}, X^{[p]}, 1,n,t^{[p]},b^{[p]}}$ and $\lambda_n^{[p]} := \lambda(A_n^{[p]})$, the largest eigenvalue of $A^{[p]}$.

THEOREM 7.2. *Let X be a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT and Φ a translation-invariant nearest neighbor interaction on X . Let t and b be globally admissible periodic rows, with common period p . Assume that $q(\Lambda_{\Phi, X}) < p_c$. Then there exist constants $Q, R > 0$ such that for sufficiently large n ,*

$$|(1/p)(\log \lambda_{n+1}^{[p]} - \log \lambda_n^{[p]}) - P(f_\Phi)| < Qe^{-Rn}.$$

Proof. Define $\Lambda^{[p]} = \Lambda_{\Phi^{[p]}, X^{[p]}}$. We will now verify that $q(\Lambda^{[p]}) < 1$. Any boundary configurations $\delta^{[p]}, \delta'^{[p]} \in \mathcal{L}_{N(0,0)}(X^{[p]})$ correspond to configurations $\delta, \delta' \in \mathcal{L}_{\partial([0,p-1] \times \{0\})}(X)$. By Proposition 5.3 (which we may use because $q(\Lambda_{\Phi, X}) < p_c < 1$), any such δ, δ' have a common globally admissible filling in X , which implies that $\delta^{[p]}, \delta'^{[p]}$ have a common globally admissible filling in $X^{[p]}$, which has positive $(\Lambda^{[p]})^{\delta^{[p]}}$, $(\Lambda^{[p]})^{\delta'^{[p]}}$ -values since $\Lambda^{[p]}$ is a Gibbs specification.

Therefore, by Proposition 3.14, for large enough n , there is a unique induced Gibbs state $\mu_{\Phi^{[p]}, X^{[p]}, 1,n,t^{[p]},b^{[p]}}$ for $X^{[p]}$ and $\Phi^{[p]}$. By again using Proposition 6.3 and (see [Kr, p. 99] and [Bl]), $\mu_{\Phi^{[p]}, X^{[p]}, 1,n,t^{[p]},b^{[p]}}$ is the unique equilibrium state for $\overline{\Phi^{[p]}}_{1,n,t^{[p]},b^{[p]}}$ on $(X^{[p]})_{1,n,t^{[p]},b^{[p]}}$ and

$$\begin{aligned} \log \lambda_n^{[p]} &= P(\overline{\Phi^{[p]}}_{1,n,t^{[p]},b^{[p]}}) = h(\mu_{\Phi^{[p]}, X^{[p]}, 1,n,t^{[p]},b^{[p]}}) \\ &\quad + \int \overline{\Phi^{[p]}}_{1,n,t^{[p]},b^{[p]}} d\mu_{\Phi^{[p]}, X^{[p]}, 1,n,t^{[p]},b^{[p]}}. \end{aligned} \tag{23}$$

We now must show that the differences of the right-hand sides of (23) for n and $n + 1$ in fact converge exponentially fast to $pP(f_\Phi)$, which is done in much the same way as in the proof of Theorem 7.1. We again take an equilibrium state μ for f_Φ and X , which is a Gibbs state for Φ on X and is the unique \mathbb{Z}^2 -MRF associated to the \mathbb{Z}^2 -specification $\Lambda_{\Phi, X}$. Since $q(\Lambda_{\Phi, X}) < 1$, we may define the unique MRF $\mu_{\Phi, X, 1,n,t,b}$ associated to $\Lambda_{\Phi, X, 1,n,t,b}$.

We claim that $\mu_{\Phi^{[p]}, X^{[p]}, 1,n,t^{[p]},b^{[p]}} = (\mu_{\Phi, X, 1,n,t,b})^{[p]}$. This follows from Proposition 3.14 and the fact that $q(\Lambda^{[p]}) < 1$, once one verifies that $(\mu_{\Phi, X, 1,n,t,b})^{[p]}$ is associated to $(\Lambda^{[p]})_{1,n,t^{[p]},b^{[p]}} = \Lambda_{\Phi^{[p]}, X^{[p]}, 1,n,t^{[p]},b^{[p]}}$. This is straightforward (but a bit tedious), and we leave the details to the reader. Thus, (23) becomes

$$\log \lambda_n^{[p]} = h((\mu_{\Phi, X, 1,n,t,b})^{[p]}) + \int \overline{\Phi^{[p]}}_{1,n,t^{[p]},b^{[p]}} d((\mu_{\Phi, X, 1,n,t,b})^{[p]}).$$

By Theorem 3.22, $h((\mu_{\Phi, X, 1,n+1,t,b})^{[p]}) - h((\mu_{\Phi, X, 1,n,t,b})^{[p]})$ converges exponentially to $ph(\mu)$. Arguing the same way as in the proof of Theorem 7.1, we see that

$$\int \overline{\Phi^{[p]}}_{1,n+1,t^{[p]},b^{[p]}} d((\mu_{\Phi, X, 1,n+1,t,b})^{[p]}) - \int \overline{\Phi^{[p]}}_{1,n,t^{[p]},b^{[p]}} d((\mu_{\Phi, X, 1,n,t,b})^{[p]})$$

converges exponentially to

$$\int f_{\Phi^{[p]}} d\mu = \int \sum_{i=0}^{p-1} (f_{\Phi} \circ \sigma_{(i,0)}) d\mu = p \int f_{\Phi} d\mu,$$

where the latter equality comes from translation-invariance of μ .

Thus, $(1/p)(\log \lambda_{n+1}^{[p]} - \log \lambda_n^{[p]})$ converges exponentially to $P(f_{\Phi})$, as desired. \square

We will make a brief aside here to consider the utility of this theorem. For a nearest neighbor \mathbb{Z}^2 -SFT X and translation-invariant nearest neighbor interaction Φ with associated Gibbs specification $\Lambda_{\Phi, X}$, recall that $\Lambda_{\Phi, X}^{\delta}$ is defined by a formula involving X and Φ for every $\delta \in \mathcal{L}_{N(0,0)}(X)$, and then, for any other $\delta' \in \mathcal{A}^{N(0,0)}$, $\Lambda_{\Phi, X}^{\delta'}$ is just defined to match one of the existing $\Lambda_{\Phi, X}^{\delta}$ (see the definition of $\Lambda_{\Phi, X}^{\delta}$ in §4). This means that $q(\Lambda_{\Phi, X})$ is, in reality, a minimum variational distance between $\Lambda_{\Phi, X}^{\delta}$ and $\Lambda_{\Phi, X}^{\delta'}$ for globally admissible δ, δ' only.

It is well known [Be] that checking whether or not a given configuration is globally admissible in a nearest neighbor \mathbb{Z}^2 -SFT can be undecidable, and so certainly algorithmically impossible. However, it is shown in [HM, Corollary 3.5] that for a strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT, global admissibility of a configuration is algorithmically checkable.

In practice, however, this checking process can be very time-consuming, and so it is often easier to consider a ‘simpler’ version of $q(\Lambda_{\Phi, X})$. Assume that Φ is defined on all of $\mathcal{A} \cup \mathcal{E}_1 \cup \mathcal{E}_2$. Say that a configuration $\delta \in \mathcal{A}^{N(0,0)}$ is *fillable* if there exists $x \in \mathcal{A}^{\{(0,0)\}} \cup N_{(0,0)}$ such that $x\delta$ is locally admissible. For fillable δ , the formula (11) makes sense and we can define

$$\widehat{q}(\Lambda_{\Phi, X}) = \max d(\Lambda_{\Phi, X}^{\delta}, \Lambda_{\Phi, X}^{\delta'})$$

where the max is taken over only fillable configurations $\delta, \delta' \in \mathcal{A}^{N(0,0)}$. Then $q(\Lambda_{\Phi, X}) \leq \widehat{q}(\Lambda_{\Phi, X})$, and so if $\widehat{q}(\Lambda_{\Phi, X}) < p_c$ then $q(\Lambda_{\Phi, X}) < p_c$ as well. Since computing $\widehat{q}(\Lambda_{\Phi, X})$ only requires finding the set of locally admissible configurations in X with shape $\{(0, 0)\} \cup N_{(0,0)}$, it is a far easier quantity to find.

8. Computability

We now address the issue of how efficiently our methods can be used to approximate $P(f_{\Phi})$ for a function f_{Φ} induced by a nearest neighbor interaction Φ on a strongly irreducible nearest neighbor SFT X .

We define $\alpha \in \mathbb{R}$ to be a *computable number* if there exists a Turing machine T which, on input n , outputs a number $p_n/q_n \in \mathbb{Q}$ such that $|\alpha - p_n/q_n| < 2^{-n}$. For any sequence of positive integers $\{r_n\}$, we say that α is $\{r_n\}$ -computable if there exists such a Turing machine T which computes p_n/q_n in less than r_n steps for all sufficiently large n . (For more information on computability theory, see [Ko].)

We say that Φ is $\{r_n\}$ -computable if each value $\Phi(w)$ in the range of Φ is $\{r_n\}$ -computable.

THEOREM 8.1. *For any $\{r_n\}$ -computable nearest neighbor Φ and strongly irreducible nearest neighbor \mathbb{Z}^2 -SFT X for which $q(\Lambda_{\Phi, X}) < p_c$, there exist constants B, C , and J such that $P(f_{\Phi})$ is $\{J^n + Br_{Cn}\}$ -computable.*

Proof. We will not include every detail of the argument, but just describe the algorithm for approximating $P(f_\Phi)$ and summarize the most computationally intensive steps. For a similar argument with more details included, see [P].

Some preprocessing must be done before any approximations. Firstly, a globally admissible periodic row t for X must be found; a careful reading of the proof in [War] shows that this can be done algorithmically. Denote by p the period of t . Also, we invest a finite number of steps to find explicit integers L and U which bound all $\Phi(w)$ from below and above respectively. This finite amount of computation is negligible compared to the computation times in the theorem, and so it can be absorbed in the constant J .

Then, by Theorem 7.2, there exists $R > 0$ such that for sufficiently large n ,

$$|1/p(\log \lambda(A_{\Phi^{[p]}, X^{[p]}, 1, n+1, t^{[p]}, t^{[p]}}) - \log \lambda(A_{\Phi^{[p]}, X^{[p]}, 1, n, t^{[p]}, t^{[p]}})) - P(f_\Phi)| < 0.2e^{-Rn}.$$

To approximate $P(f_\Phi)$ to within 2^{-n} , it then clearly suffices to approximate

$$\lambda(A_{\Phi^{[p]}, X^{[p]}, 1, k, t^{[p]}, t^{[p]}}) \quad \text{and} \quad \lambda(A_{\Phi^{[p]}, X^{[p]}, 1, k+1, t^{[p]}, t^{[p]}})$$

to within $0.4 \cdot 2^{-n}$, where $k = n \lceil \log 2/R \rceil$. It obviously suffices to describe the procedure for $\lambda(A_{\Phi^{[p]}, X^{[p]}, 1, k, t^{[p]}, t^{[p]}})$. We from now on refer to $A_{\Phi^{[p]}, X^{[p]}, 1, k, t^{[p]}, t^{[p]}}$ simply as A for ease of reading.

The entries of A , indexed by legal columns c and d , are all of the form $e^{-\sum \Phi(w)}$, where the sum is always over a set of at most $4kp$ configurations w on single sites or pairs of adjacent sites; these are easily computed in polynomial (negligible) time in k , given t , c , and d . Therefore, the smallest non-zero entry of A is at least e^{-4kpU} and the largest entry is at most e^{-4kpL} . We now wish to approximate each entry of A to within a tolerance of $0.2|\mathcal{A}|^{-kp} e^{-4kpU} e^{8kpL} 2^{-n}$. We first approximate each individual $\Phi(w)$ to within $1/40kp|\mathcal{A}|^{-kp} e^{-4kpU} e^{8kpL} 2^{-n}$. Since k is linear in n , this expression is only exponentially small in n . Since each $\Phi(w)$ is $\{r_n\}$ -computable, there exists C so that such an approximation can be found for each $\Phi(w)$ in fewer than r_{Cn} steps for sufficiently large n , and so a collection of such approximations for all $\Phi(w)$ can be found in fewer than Br_{Cn} steps, where B is the constant number of configurations w for which $\Phi(w) \neq 0$. For each entry $A_{c,d} = e^{-\sum \Phi(w)}$ of A , we then have an approximation to $-\sum \Phi(w)$ within a tolerance of $0.1|\mathcal{A}|^{-kp} e^{-4kpU} e^{8kpL} 2^{-n}$, which yields an approximation to $A_{c,d} = e^{-\sum \Phi(w)}$ to within a tolerance of $0.2|\mathcal{A}|^{-kp} e^{-4kpU} e^{4kpL} 2^{-n}$ for large enough n since $A_{c,d} < e^{-4kpL}$.

We then have a matrix \tilde{A} in which each entry is within $0.2|\mathcal{A}|^{-kp} e^{-4kpU} e^{4kpL} 2^{-n}$ of the corresponding entry of A . Since this matrix has only exponentially many entries (the size of $A = A_{\Phi^{[p]}, X^{[p]}, 1, k, t^{[p]}, t^{[p]}}$ is at most the size of the alphabet of $X_{1, k, t^{[p]}, t^{[p]}}^{[p]}$, which is at most $|\mathcal{A}|^{kp}$), and since each approximation only involves summing previously recorded approximations to $\Phi(w)$ and exponentiating, there exists F so that \tilde{A} can be computed in fewer than $F^n + Br_{Cn}$ steps for sufficiently large n .

Then

$$(1 - 0.2|\mathcal{A}|^{-kp} e^{4kpL} 2^{-n})A < \tilde{A} < (1 + 0.2|\mathcal{A}|^{-kp} e^{4kpL} 2^{-n})A,$$

and by monotonicity of the Perron eigenvalue,

$$(1 - 0.2|\mathcal{A}|^{-kp} e^{4kpL} 2^{-n})\lambda(A) < \lambda(\tilde{A}) < (1 + 0.2|\mathcal{A}|^{-kp} e^{4kpL} 2^{-n})\lambda(A).$$

Since clearly $\lambda(A)$ is bounded from above by the maximum row sum of A , which itself is less than $|\mathcal{A}|^{kp} e^{-4kpL}$, this means that $|\lambda(\tilde{A}) - \lambda(A)| < 0.2 \cdot 2^{-n}$.

All that remains is to approximate $\lambda(\tilde{A})$ to within a tolerance of $0.2 \cdot 2^{-n}$. Since \tilde{A} and A have the same non-zero entries, and since A is primitive, there exists $N = N(k)$ such that \tilde{A}^N has all positive entries. We assume that N is the smallest such integer, which is called the index of primitivity of A . It is well known [HorJ, Corollary 8.5.9] that the index of primitivity is at most quadratic in the size of the matrix, so there exists G independent of n so that $N < G^k$. Let ϵ denote the smallest entry of \tilde{A}^N . Then $\epsilon \geq (e^{-5kpU})^N$. (We changed $4kpU$ to $5kpU$ to account for the fact that the smallest entry of \tilde{A} could be slightly smaller than the smallest entry of A .)

The reader can verify that for any M and k ,

$$\left(\epsilon \sum (\tilde{A}^M)_{c,d} \right)^k < \sum (\tilde{A}^{k(M+N)})_{c,d} < \left(\sum (\tilde{A}^{M+N})_{c,d} \right)^k.$$

By taking logs, dividing by $k(M + N)$, and letting $k \rightarrow \infty$, we see that

$$\frac{\log \epsilon}{M + N} + \frac{\log \sum (\tilde{A}^M)_{c,d}}{M + N} < \lambda(\tilde{A}) < \frac{\log \sum (\tilde{A}^{M+N})_{c,d}}{M + N}.$$

If we write

$$f_M = \frac{\log \sum (\tilde{A}^M)_{c,d}}{M},$$

then, for every $M > N$,

$$\lambda(\tilde{A}) < f_M < \lambda(\tilde{A}) + \frac{N\lambda(\tilde{A})}{M} - \frac{\log \epsilon}{M}.$$

For f_M to approximate $\lambda(\tilde{A})$ to within $0.2 \cdot 2^{-n}$, it is therefore sufficient to take $M > 5 \cdot 2^n (N\lambda(\tilde{A}) - \log \epsilon)$, which is less than H^n for some constant H and large enough n . The calculation of f_{H^n} entails taking an exponentially large power of an exponentially large matrix, which can be done in exponentially many computations. Therefore, there exists I so that $\lambda(\tilde{A})$ can be approximated to within $0.2 \cdot 2^{-n}$ in fewer than I^n computations.

By collecting all of these facts and taking $J = \max(I, F) + 1$, we see that, for sufficiently large n , we may approximate $P(f_\Phi)$ to within 2^{-n} by performing fewer than $J^n + Br_{C_n}$ steps for some uniform constants B, C , and J . □

9. Examples

We now give some applications of the results from §§7 and 8 to specific Gibbs states and pressures, beginning with the hard-core and Ising antiferromagnetic models presented earlier. We note that these models have strongly irreducible underlying SFTs (the hard square shift and full shift on ± 1 respectively), and so checking whether our results apply to them boils down to checking which parameter values give $q(\Lambda_{\Phi, X}) < p_c$.

(1) Hard-core model. We wish to know which activity levels a give $q(\Lambda_{\Phi, X}) < p_c$. By the definition of Φ , it is easy to check that for $\delta = 0^{N(0,0)}$,

$$\Lambda_{\Phi, \mathcal{H}}^\delta(b) = \begin{cases} \frac{1}{1+a} & \text{if } b = 0, \\ \frac{a}{1+a} & \text{if } b = 1, \end{cases}$$

which we abbreviate by $\Lambda_{\Phi, \mathcal{H}}^\delta = (1/(1+a), a/(1+a))$. For any other $\delta' \in \{0, 1\}^{N(0,0)}$, $\Lambda_{\Phi, \mathcal{H}}^{\delta'} = (1, 0)$, i.e. it is concentrated entirely on the 0 symbol. (This is because, if $\delta' \neq 0^{N(0,0)}$, the only locally admissible way to fill δ in \mathcal{H} is with a 0.) Therefore,

$$q(\Lambda_{\Phi, \mathcal{H}}) = d\left(\left(\frac{1}{1+a}, \frac{a}{1+a}\right), (1, 0)\right) = \frac{a}{1+a},$$

which is less than p_c if and only if $a < p_c/(1 - p_c)$. We note that this computation was already done in [BM].

(2) Ising antiferromagnet. Again, we wish to know which parameters h, β give $q(\Lambda_{\Phi, X}) < p_c$. Again, it is reasonably straightforward to check that for any $\delta \in \{\pm 1\}^{N(0,0)}$ with $\sum_{v \in N(0,0)} \delta(v) = n$,

$$\Lambda_{\Phi, X}^\delta(b) = \begin{cases} \frac{\exp(-\beta(h-n))}{\exp(\beta(h-n)) + \exp(-\beta(h-n))} & \text{if } b = -1, \\ \frac{\exp(\beta(h-n))}{\exp(\beta(h-n)) + \exp(-\beta(h-n))} & \text{if } b = 1. \end{cases}$$

It is then fairly easy to see that

$$\begin{aligned} q(\Lambda_{\Phi, X}) &= d((\Lambda_{\Phi, X})^{(1^{N(0,0)})}, (\Lambda_{\Phi, X})^{(-1^{N(0,0)})}) \\ &= \frac{\exp(\beta(h-4))}{\exp(\beta(h-4)) + \exp(-\beta(h-4))} - \frac{\exp(\beta(h+4))}{\exp(\beta(h+4)) + \exp(-\beta(h+4))}. \end{aligned}$$

It was shown in [BM] that $q(\Lambda_{\Phi, X}) < p_c$ as long as

$$2\beta(4 - |h|) < \log\left(\frac{p_c}{1 - p_c}\right),$$

though this condition is certainly not necessary.

In both models, the results of [BM] imply that there is a unique Gibbs measure μ for the described parameters. Theorem 7.2 implies that in addition, the pressure $P(f_\Phi)$ is exponentially well approximable by

$$1/p(\log \lambda(A_{\Phi^{[p]}, X^{[p]}, 1, n+1, t^{[p]}, b^{[p]}}) - \log \lambda(A_{\Phi^{[p]}, X^{[p]}, 1, n, t^{[p]}, b^{[p]}}))$$

for any boundary conditions t, b periodic with period p . (Of course, for these two models, there exist globally admissible constant rows, and so we could take $p = 1$.) By Theorem 8.1, a bound on the computability of these pressures can also be given in terms of the computability of the relevant parameter values.

We conclude by giving applications to topological entropy, in the same spirit as [P].

Definition 9.1. Let X be a nearest neighbor \mathbb{Z}^2 -SFT. Define the specification $\Lambda = \Lambda_{0, X}$ by

$$\Lambda^\delta(\eta) := \begin{cases} 0 & \text{if } \eta\delta \notin \mathcal{L}(X), \\ \frac{1}{N(\delta)} & \text{if } \eta\delta \in \mathcal{L}(X), \end{cases}$$

when $\delta \in \mathcal{L}(X)$ (here $N(\delta)$ is just the normalization factor $|\{\eta \in \mathcal{A}^S : \eta\delta \in \mathcal{L}(X)\}|$), and by $\Lambda^{\delta'} = \Lambda^{\delta_0}$ when $\delta' \notin \mathcal{L}(X)$, where δ_0 is any fixed globally admissible boundary with the same shape as δ' .

Note that since a measure of maximal entropy is clearly an equilibrium state for the function $f = f_0 = 0$, any measure of maximal entropy is a \mathbb{Z}^2 -MRF associated to the specification $\Lambda_{0,X}$.

We now exhibit two classes of strongly irreducible nearest neighbor \mathbb{Z}^2 -SFTs for which $q(\Lambda) < p_c$ for this specification, implying that the topological entropy (topological pressure for $f = f_0 = 0$) is exponentially well approximable via strips, and therefore $\{J^n\}$ -computable for some J . For the first, we need a definition.

Definition 9.2. In a nearest neighbor \mathbb{Z}^d -SFT X with alphabet \mathcal{A} , $a \in \mathcal{A}$ is a safe symbol if for all $b \in \mathcal{A}$, $(a, b), (b, a) \in \mathcal{E}_i$ for $1 \leq i \leq d$. In other words, a is a safe symbol if it may legally appear next to any letter of the alphabet in any direction.

PROPOSITION 9.3. Any nearest neighbor \mathbb{Z}^2 -SFT with alphabet \mathcal{A} containing a subset \mathcal{A}' of safe symbols for which $|\mathcal{A}'|/|\mathcal{A}| > 1 - p_c$ has a unique measure of maximal entropy μ which is an MRF associated to a translation-invariant \mathbb{Z}^2 -specification Λ satisfying $q(\Lambda) < p_c$. The topological entropy of any such SFT is $\{J^n\}$ -computable for some J .

Proof. Consider any such \mathbb{Z}^2 -SFT X and any μ a measure of maximal entropy on X with \mathbb{Z}^2 -specification Λ . Clearly, since X contains a safe symbol, any $\delta \in \mathcal{A}^{N(0,0)}$ is in $\mathcal{L}(X)$ and X is strongly irreducible. For any $\delta \in \mathcal{A}^{N(0,0)}$, the probability distribution Λ^δ is uniform over some subset of \mathcal{A} , call it S_δ , which contains \mathcal{A}' . Clearly, $|S_\delta| = N(\delta)$. Choose any $\delta, \delta' \in \mathcal{A}^{N(0,0)}$, and assume without loss of generality that $N(\delta) \geq N(\delta')$. Then,

$$\begin{aligned} d(\Lambda^\delta, \Lambda^{\delta'}) &= \frac{1}{2} \sum_{e \in \mathcal{A}} |\Lambda^\delta(e) - \Lambda^{\delta'}(e)| \\ &\leq \frac{|\mathcal{A}'|}{2} \left(\frac{1}{N(\delta')} - \frac{1}{N(\delta)} \right) + \frac{N(\delta) - |\mathcal{A}'|}{2N(\delta)} + \frac{N(\delta') - |\mathcal{A}'|}{2N(\delta')} \\ &= 1 - \frac{|\mathcal{A}'|}{N(\delta)} \leq 1 - \frac{|\mathcal{A}'|}{|\mathcal{A}|} < p_c. \end{aligned}$$

Therefore, $q(\Lambda) < p_c$, implying by Theorem 3.9 that μ was unique. The fact that $h(X) = P(f_0)$ implies that $h(X)$ is $\{J^n\}$ -computable for some J by Theorem 8.1. \square

We note that since $p_c > 0.556 > 0.5$ by [BE], clearly the \mathbb{Z}^2 hard square shift \mathcal{H} satisfies the conditions of Proposition 9.3.

We recall that the $\{J^n\}$ -computability of the topological entropy of such SFTs followed from the exponentially good approximations given by differences of topological entropies of constrained strips, i.e. strips with boundary conditions t, b . For SFTs with at least one safe symbol a (such as those to which Proposition 9.3 applies), one can take $t, b = a^{\mathbb{Z}}$ and then the topological entropies of the approximating ‘constrained’ strips are equal to those of unconstrained strips as were treated in [P].

PROPOSITION 9.4. Let X be a nearest neighbor \mathbb{Z}^2 -SFT with alphabet \mathcal{A} with the property that for all $a \in \mathcal{A}$, and for any direction, the set of legal neighbors of a in that direction has cardinality greater than $(1 - (p_c/4(1 + p_c)))|\mathcal{A}|$. Then X has a unique measure of maximal entropy μ which is an MRF associated to a \mathbb{Z}^2 -specification Λ satisfying $q(\Lambda) < p_c$. The topological entropy of any such SFT is $\{J^n\}$ -computable for some J .

Proof. Consider any such nearest neighbor \mathbb{Z}^2 -SFT X and any μ a measure of maximal entropy on X with \mathbb{Z}^2 -specification Λ . Since $1 - (p_c/4(1 + p_c)) > \frac{3}{4}$, at any site v , the presence of a symbol at an adjacent site rules out fewer than $(1/4)|\mathcal{A}|$ of the symbols at v ; thus, any $\delta \in \mathcal{A}^{N(0,0)}$ can be extended to a locally admissible configuration with shape $N_{(0,0)} \cup \{0\}$. It follows that, any locally admissible configuration x on a subset S of \mathbb{Z}^2 can be extended to a locally admissible configuration on $S \cup \{v\}$ for any site $v \notin S$. Therefore, any locally admissible configuration is globally admissible. This implies both that any $\delta \in \mathcal{A}^{N(0,0)}$ is in $\mathcal{L}(X)$ and that X is strongly irreducible (with filling distance 1). Therefore, for any $\delta \in \mathcal{A}^{N(0,0)}$, the probability distribution Λ^δ is uniform over some non-empty subset of \mathcal{A} , call it S_δ . Clearly $|S_\delta| = N(\delta)$. Define $\alpha := p_c/(1 + p_c)$. Then it is clear that $|S_\delta| > (1 - \alpha)|\mathcal{A}|$ for any $\delta \in \mathcal{A}$. Choose any $\delta, \delta' \in \mathcal{A}^{N(0,0)}$, and assume without loss of generality that $N(\delta) \geq N(\delta')$. Define $m := |S_\delta \cap S_{\delta'}|$ and note that $m > N(\delta) - \alpha|\mathcal{A}|$. Then,

$$\begin{aligned} d(\Lambda^\delta, \Lambda^{\delta'}) &= \frac{1}{2} \sum_{e \in \mathcal{A}} |\Lambda^\delta(e) - \Lambda^{\delta'}(e)| \\ &\leq \frac{m}{2} \left(\frac{1}{N(\delta')} - \frac{1}{N(\delta)} \right) + \frac{N(\delta) - m}{2N(\delta)} + \frac{N(\delta') - m}{2N(\delta')} \\ &= 1 - \frac{m}{N(\delta)} < 1 - \frac{N(\delta) - \alpha|\mathcal{A}|}{N(\delta)} = \frac{\alpha|\mathcal{A}|}{N(\delta)} < \frac{\alpha}{1 - \alpha} = p_c. \end{aligned}$$

Therefore, $q(\Lambda) < p_c$, implying by Theorem 3.9 that μ was unique. The fact that $h(X) = P(f_0)$ again implies that $h(X)$ is $\{J^n\}$ -computable for some J by Theorem 8.1. \square

By [BE],

$$\frac{p_c}{4(1 + p_c)} > \frac{0.556}{4(1 + 0.556)} > \frac{5}{56}.$$

It follows that the usual k -checkerboard \mathbb{Z}^2 -SFT, with alphabet $\{1, \dots, k\}$ and forbidden list consisting of all pairs of adjacent identical letters, satisfies the conditions of Proposition 9.4 when $k \geq 12$. (It may also satisfy these conditions for smaller k depending on the exact value of p_c .)

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