Review Questions Solutions

Question 1

$$\frac{dy}{dx} + ycos(x) = 4cos(x)$$

$$\frac{dy}{dx} = cos(x)(4-y)$$

$$\boxed{\text{TIP: treat } \frac{dy}{dx} \text{ like a fraction}}$$

$$\frac{1}{4-y}dy = cos(x)dx$$

$$\int \frac{1}{4-y}dy = \int cos(x)dx$$

$$-ln(4-y) = sin(x) + C$$

$$4 - y = e^{-sin(x)-C}$$

Let $C^* = e^{-C}$

$$y = 4 - (C^*)(e^{sin(x)})$$

$y(0) = 6 \rightarrow$

$$6 = 4 - C^*$$

$$C^* = -2$$

Therefore

$$y = 4 + 2e^{-sin(x)}$$

Question 2

$$y' + 5y = e^{4x}$$

since it isn't separable, use integrating factor

$$\mu = e^{\int_0^x 5dt} = e^{5x}$$

TIP:
given a first order linear ODE

$$a(x)y'(x) + b(x)y(x) = c(x)$$

with initial condition x_0

$$y'(x) + p(x)y(x) = q(x)$$

where $p(x) = b(x)/a(x), q(x) = c(x)/a(x)$

then the integrating factor is:

$$\mu = e^{\int_{x_o}^{x} p(t)dt}$$

Therefore

$$e^{5x}(y' + 5y) = e^{5x}e^{4x}$$
$$\frac{d}{dx}(e^{5x}y) = e^{9x}$$
$$e^{5x}y = \int_0^x e^{9x}dx$$
$$e^{5x}y = \frac{1}{9}e^{9x} - \frac{1}{9} + C$$
$$y = \frac{1}{9}e^{4x} - \frac{1}{9}e^{-5x} + Ce^{-5x}$$

$y(0) = 3 \rightarrow$

$$3 = \frac{1}{9} - \frac{1}{9} + C$$
$$C = 3$$

Therefore

$$y = \frac{1}{9}e^{4x} + (3 - \frac{1}{9})e^{-5x}$$
$$y = \frac{1}{9}e^{4x} + \frac{26}{9}e^{-5x}$$

Question 3

$$\lambda = \frac{-10 \pm \sqrt{100 - (4 \times 25)}}{2}$$

$$\lambda = -5$$

Therefore

$$y = C_1 e^{-5t} + C_2 t e^{-5t}$$

TIP:
For constant coefficient second order linear ODE's , there are 3 cases based on the roots.

CASE 1:$\lambda^{\pm}$ real, $\lambda^{+} \neq \lambda^{-}$

$$y = C_1 e^{\lambda^{+} t} + C_2 e^{\lambda^{-} t}$$

CASE 2: $\lambda^{+} = \lambda^{-} = \lambda$

$$y = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$$

CASE 3: $\lambda^{\pm} = \mu \pm \omega i$ (complex)

$$y = C_1 e^{\mu t} cos(\omega t) + C_2 e^{\mu t} sin(\omega t)$$

$y(1) = 0$ $y'(1) = 1$

$$y(1) = C_1 e^{-5} + C_2 e^{-5} = e^{-5}(C_1 + C_2) = 0$$
$$C_2 = -C_1$$

$y' = e^{-5t}(-5C_1 + C_2(1-5))$

$$y'(1) = e^{-5}(-5C_1 - 4C_2) = -e^{-5} C_1 = 1$$

Therefore $C_1 = -e^5, C_2 = e^5$

$$y = -e^5 e^{-5t} + e^5 t e^{-5t}$$

Question 4
$x(0) = 11$ and $y(0) = -9$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -10 & -12 \\ 9 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

SHORTCUT:
for a 2x2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the characteristic polynomial is:

$$\lambda^2 - tr(\mathbf{A}) + det(\mathbf{A}) = 0$$

where trace: $tr(\mathbf{A}) = a + d$, determinant: $det(\mathbf{A}) = ad - bc$

$trace = -10 + 11 = 1$
$determinant = (-10 \times 11) - (9 \times -12) = -2$

$$\lambda^2 - \lambda - 2 = 0$$

Therefore $\lambda = 2, -1$

For $\lambda = 2 : \mathbf{A} - \lambda\mathbf{I}$

$$\begin{bmatrix} -12 & -12 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-12v_1 - 12v_2 = 0 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = -1 : \mathbf{A} - \lambda\mathbf{I}$

$$\begin{bmatrix} -9 & -12 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-9v_1 - 12v_2 = 0 \rightarrow \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$\mathbf{P} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$x_0 = \begin{bmatrix} 11 \\ -9 \end{bmatrix}$

SHORTCUT: Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then:

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{P}e^{\mathbf{D}t} = \begin{bmatrix} e^{2t} & 4e^{-t} \\ -e^{2t} & -3e^{-t} \end{bmatrix}$$

$$\mathbf{P}^{-1}\underline{x_0} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Therefore the solution $\underline{x(t)} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$ is:

$$3 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} + 2 \begin{bmatrix} 4e^{-t} \\ -3e^{-t} \end{bmatrix}$$

Therefore

$$x(t) = 3e^{2t} + 8e^{-t}$$
$$y(t) = -3e^{2t} - 6e^{-t}$$

Question 5

1.Given $\frac{dT}{dt} = k(T - T_s)$ where $T_s = 18$ is a constant
$T(0) = 95$
when $T = 70$, $\frac{dT}{dt} = -2$

$$-2 = k(70 - 18)$$
$$k = -\frac{1}{26}$$

2.By inspection, we can see that

$$\frac{dT}{dt} = k(T - T_s) = -\frac{1}{26}(T - T_s) = 0$$

only when $T = T_s = 18$

3.

TIP:
Most differential equations cannot be solved exactly/analytically, so we use methods (i.e. Euler's Method) to approximate solutions.
Suppose y_i is an approximation to $y(t_i)$ Then y_{i+1} is:
$$y_{i+1} = y_i + f(y_i, t_i)(t_{i+1} - t_i)$$

Using Euler's with h=2, for a total length of 10 minutes.
We are given that $T(0) = 95$.
$T(2) = T(0) + 2T'(0) = 95 - \frac{2}{26}(95 - 18) \approx 89.0769$
$T(4) = T(2) + 2T'(2) = 89.0769 - \frac{2}{26}(89.0769 - 18) \approx 83.6094$
$T(6) = T(4) + 2T'(4) = 83.6094 - \frac{2}{26}(83.6094 - 18) \approx 78.5625$
$T(8) = T(6) + 2T'(6) = 78.5625 - \frac{2}{26}(78.5625 - 18) \approx 73.9039$
$T(10) = T(8) + 2T'(8) = 73.9039 - \frac{2}{26}(73.9039 - 18) \approx 69.60$

Question 6

Taking Laplace transform of both sides of the equation and taking the initial conditions into consideration, we obtain the transformed ODE

$$s^2Y(s) + 2sY(s) + 2Y(s) = \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s},$$

so that

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-\frac{\pi}{2}s}}{s^2+2s+2}.$$

Using partial fractions

$$Y_1(s) = \frac{s}{(s^2+1)(s^2+2s+2))} = \frac{1}{5}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right].$$

We can also write

$$\frac{s+4}{s^2+2s+2} = \frac{(s+1)+3}{(s+1)^2+1},$$

therefore

$$\mathcal{L}^{-1}\big[Y_1(s)\big] = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\big[\cos t + 3\sin t\big].$$

On the other hand,

$$\mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi}{2}s}}{s^2+2s+2}\right] = e^{-\left(t-\frac{\pi}{2}\right)}\sin\left(t-\frac{\pi}{2}\right)u_{\frac{\pi}{2}}(t).$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\big[\cos t + 3\sin t\big] - e^{-\left(t-\frac{\pi}{2}\right)}\cos t\, u_{\frac{\pi}{2}}(t).$$

Question 7
Set $x_1 = u$, $x_2 = u'$, $x_3 = u$", $x_4 = u'''$, so that the given equation takes the form

$$x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4, \quad x_4' = x_1.$$

This is represented by the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{g}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin t \end{pmatrix}.$$

Question 8
The characteristic equation of the coefficient matrix is $r^3 - 3r^2 + 3r - 1 = 0$, with a single root $r = 1$ of multiplicity 3. Setting $r = 1$, we obtain the eigenvalue equation

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system of linear equations reduces to single equation

$$4\xi_1 - 3\xi_2 - 2\xi_3 = 0.$$

Since the equation has two free variables, we have two linearly independent eigenvectors, for instance

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}.$$

Therefore two linearly independent solutions are obtained as

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t \quad \text{and} \quad \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^t.$$

To find a third solution, we try a function of the form $\mathbf{x} = \boldsymbol{\xi} te^t + \boldsymbol{\eta} e^t$. It follows that

$$\mathbf{x}' = \boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t.$$

Hence the coefficient vectors must satisfy $\boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t = \mathbf{A}\boldsymbol{\xi} te^t + \mathbf{A}\boldsymbol{\eta} e^t$. Rearranging the terms we have

$$\boldsymbol{\xi} e^t = (\mathbf{A} - \mathbf{I})\boldsymbol{\xi} te^t + (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} e^t.$$

Matching coefficients, it follows that $(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ and $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$. Thus $\boldsymbol{\xi}$ is an eigenvector A, i.e.

$$\boldsymbol{\xi} = \alpha \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \alpha \\ 2\beta \\ 2\alpha - 3\beta \end{pmatrix}.$$

The system of equations $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ then reduces to

$$4\eta_1 - 3\eta_2 - 2\eta_3 = \alpha, \quad 8\eta_1 - 6\eta_2 - 4\eta_3 = 2\beta, \quad -4\eta_1 + 3\eta_2 + 2\eta_3 = 2\alpha - 3\beta.$$

This is consistent provided $\alpha = \beta$. We have to be careful in choosing a value of $\alpha = \beta$ and the free variables η_1, η_2 of the resulting equation; for instance choosing $\alpha = \beta = 0$ would result in $\mathbf{x}$ being linearly dependent on $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. A convenient choice is $\alpha = \beta = -2$ and $\eta_1 = \eta_2 = 0$, $\eta_3 = 1$. Therefore the third linearly independent solution is

$$\mathbf{x}^{(3)} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t.$$

Question 9

(a) The critical points are the solutions of the system

$$x(a - \sigma x - \alpha y) = 0, \qquad y(-c + \gamma x) = 0.$$

If $x = 0$, then $y = 0$. If $y = 0$, then $x = \frac{a}{\sigma}$. The third solution is found by substituting $x = c/\gamma$ into the first equation. This implies that $y = a/\alpha - \sigma c/(\gamma\alpha)$. So the critical points are $(0,0)$, $(\frac{a}{\sigma}, 0)$ and $(\frac{c}{\gamma}, \frac{a}{\gamma} - \frac{\sigma c}{\gamma\alpha})$. When σ is increasing, the critical point $(\frac{a}{\sigma}, 0)$ moves to the left and the critical point $(\frac{c}{\gamma}, \frac{a}{\gamma} - \frac{\sigma c}{\gamma\alpha})$ moves down. The assumption $a > \frac{\sigma c}{\gamma}$ is necessary for the third critical point to be in the first quadrant.

(b,c) The Jacobian of the system is

$$\mathbf{J} = \begin{pmatrix} a - 2\sigma x - \alpha y & -\alpha x \\ \gamma y & -c + \gamma x \end{pmatrix}.$$

This implies that at the origin

$$\mathbf{J}(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix},$$

which implies that the origin is a saddle point (since $a > 0$ and $c > 0$ by our assumption).

At the critical point $(\frac{a}{\sigma}, 0)$

$$\mathbf{J}(\frac{a}{\sigma}, 0) = \begin{pmatrix} -a & -\frac{\alpha a}{\sigma} \\ 0 & -c + \frac{\gamma a}{\sigma} \end{pmatrix},$$

which implies that this critical point is also a saddle as long as our assumption $a > \frac{\sigma c}{\gamma}$ is valid.

At the critical point $(\frac{c}{\gamma}, \frac{a}{\alpha} - \frac{\sigma c}{\gamma\alpha})$,

$$\mathbf{J}\left(\frac{c}{\gamma}, \frac{a}{\alpha} - \frac{\sigma c}{\gamma\alpha}\right) = \begin{pmatrix} -\frac{\sigma c}{\gamma} & -\frac{\alpha c}{\gamma} \\ \frac{\gamma a}{\alpha} - \frac{\sigma c}{\alpha} & 0 \end{pmatrix}.$$

The eigenvalues of the matrix are

$$\frac{-c\sigma \pm \sqrt{c^2\sigma^2 + 4c^2\gamma\sigma - 4ac\gamma^2}}{2\gamma}$$

We set the discriminant equal to zero and find that the greater solution is

$$\sigma_1 = -2\gamma + \frac{2\gamma}{c}\sqrt{ac + c^2}.$$

First note that $\sigma_1 > 0$, since $\sqrt{ac + c^2} > c$. Next we note that $\sigma_1 < \frac{a\gamma}{c}$. Since

$$\sqrt{ac + c^2} < \sqrt{\frac{a^2}{4} + ac + c^2} = \frac{a}{2} + c,$$

we see that

$$\sigma_1 = -2\gamma + \frac{2\gamma}{c}\sqrt{ac + c^2} < -2\gamma + \frac{2\gamma}{c}\left(\frac{a}{2} + c\right) = -2\gamma + \frac{a\gamma}{c} + 2\gamma = \frac{a\gamma}{c}.$$

For $0 < \sigma < \sigma_1$, the eigenvalues will be complex conjugates with negative real part, so the critical point will be an asymptotically stable spiral point. For $\sigma = \sigma_1$, the eigenvalues will be repeated and negative, so the critical point will be an asymptotically stable spiral point or node. For $\sigma_1 < \sigma < \frac{ac}{\gamma}$, the eigenvalues will be distinct and negative, so the critical point will be an asymptotically stable node.

(d) Since the third critical point is asymptotically stable for $0 < \sigma < \frac{ac}{\gamma}$, and the other critical points are saddle points, the populations will coexist for all such values of σ.