MATH 215/255 QUIZ 1: SOLUTIONS (SUMMER 2015, T1)

QUESTION 1

(i) Solve:

$$\frac{dy}{dt} = y - y^2$$
$$\frac{dy}{y(1-y)} = dt$$

Partial Fraction expansion

$$(\frac{1}{y} + \frac{1}{(1-y)})dy = dt$$

Integrate of both sides

$$\int (\frac{1}{y} + \frac{1}{(1-y)})dy = \int dt$$
$$\ln|y| - \ln|1-y| = t + C$$
$$\ln\frac{|y|}{|1-y|} = t + C$$
$$\frac{|y|}{|1-y|} = Ae^t$$
$$where, A = e^C > 0$$

Eliminate absolute value.

$$y(0) = \frac{1}{2} < 1$$
$$pick : \frac{y}{1-y}$$

Determine "A"

$$\frac{y(0)}{1-y(0)} = Ae^0$$
$$1 = A$$

which simplifies to ,

$$\frac{y}{1-y} = e^t$$
$$\underline{\underline{y(t) = \frac{e^t}{1+e^t}}}$$

(ii) Limiting behaviour as $t \to \infty$ (2 pts)

Rewrite y(t):

$$y(t) = \frac{1}{1+e^{-t}}$$

As $t \to \infty$ then $e^{-t} \to 0$. So,

$$\underline{\underline{\lim_{t\to\infty} y(t) = 1}}$$

QUESTION 2

Let the amount of salt in side the tank be denoted by S(t) [in grams] and the concentration of the Brine be A g/l. Since the inflow rate (1 l/min) = outflow rate (1 l/min), volume of the solution in the tank remains constant (20 liters)

(i) Setup differential equation:

$$\frac{ds}{dt} = A(\frac{g}{l}) \times 1(\frac{l}{min}) - \frac{s(t)}{20} \times 1(\frac{l}{min})$$

$$\frac{ds}{dt} = A - \frac{s(t)}{20}$$

$$\frac{ds}{dt} + \frac{1}{20}s(t) = A$$

(ii) Find integrating factor: where $p(x) = \frac{1}{20}$

$$r(t) = e^{\int \frac{1}{20}dt}$$

$$r(t) = e^{\frac{t}{20}}$$

(iii) Multiply both sides by the integrating factor and solve the equation:

$$[s(t)e^{\frac{t}{20}}]' = Ae^{\frac{t}{20}}$$

$$s(t)e^{\frac{t}{20}} = 20Ae^{\frac{t}{20}} + C(constant)$$

$$s(t) = 20A + \frac{C}{e^{\frac{t}{20}}}$$

(iv) Plug in initial condition and the amount of salt present at the end of the 20 minutes:

$$5 = 20A + C \tag{1}$$

$$15 = 20A + \frac{C}{e} \tag{2}$$

(v) Multiple (2) by "e" and subtract (1):

$$15e - 5 = 20A(e-1)$$
$$\underline{\underline{A = \frac{1}{4}(\frac{3e-1}{e-1})}}(\frac{g}{l})$$

QUESTION 3

(a) Find y(h) and y(2h)

$$y(h) = y(0) + y'(0) \times h$$

Given $y' = y$ & $y(0) = 1$, we will have

$$y(h) = 1 + h$$

Similarly

$$y(2h) = y(h) + y'(h) \times h$$
$$y(2h) = 1 + h + (1+h) \times h$$
$$y(2h) = (1+h)^2$$

(b) From the above solutions we can see that, for n= 1,2,3 .....

$$y(nh) = (1+h)^n$$

Fix $x = nh_n$, such that $y(nh) = y_n(x)$ and $h_n = \frac{x}{n}$

$$y_n(x) = (1 + \frac{x}{n})^n$$

If we let $n \to \infty$, we get

$$y(x) = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$
$$\underline{\underline{y(x) = e^x}}$$