Midterm 1 Review: Practice Problems Solution Hints

The title says it all. These are hints only. Use them to write out complete solutions.

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1. We used the following lemma in the proof of the Stone-Weierstrass theorem. Prove it.

   "Let $\mathcal{A}$ be an algebra of real-valued functions on some set X and suppose that $\mathcal{A}$ separates points and vanishes at no point of X. Then given any two points x_0, y_0 in X, $x_0 \neq y_0$ and $a, b \in \mathbb{R}$, we can find a function $f \in \mathcal{A}$ with $f(x_0) = a$ and $f(y_0) = b$."

   Keep in mind that X need not be compact and the functions in $\mathcal{A}$ need not be continuous.

   Hint. Since $\mathcal{A}$ separates points and vanishes at no point, there exist $g, h, k \in \mathcal{A}$ such that $g(x_0) \neq g(y_0)$, $h(x_0) \neq 0$, $k(y_0) \neq 0$. Define

   $$u(x) = [g(x) - g(y_0)]\, h(x) \qquad v(x) = [g(x) - g(x_0)]\, k(x).$$

   Then $u, v \in \mathcal{A}$ (why?), with $u(x_0) \neq 0, u(y_0) = 0$, and $v(x_0) = 0, v(y_0) \neq 0$. Check that the function f given by

   $$f(x) = a\frac{u(x)}{u(x_0)} + b\frac{v(x)}{v(y_0)}$$

   lies in $\mathcal{A}$ and obeys all the desired properties. □

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2. The proof of Stone-Weierstrass theorem also involved the following proposition. Prove it.

   "*Given any metric space X, let $\mathcal{A} \subseteq B(X;\mathbb{R})$ denote a subalgebra of the space of bounded real-valued functions on $\mathbb{R}$. Show that the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ is both a subalgebra and a sublattice (i.e., if $f \in \overline{\mathcal{A}}$, then $|f| \in \overline{\mathcal{A}}$).*"

   Verify also that the same statement holds with $B(X;\mathbb{R})$ replaced by $C_b(X;\mathbb{R})$, the space of bounded, continuous real-valued functions on X.

   Hint. It is up to you to check that $\overline{\mathcal{A}}$ is a subalgebra. We will verify that $\overline{\mathcal{A}}$ is a sublattice: given $f \in \overline{\mathcal{A}}$, we have $|f| \in \overline{\mathcal{A}}$. In other words, for any $\epsilon > 0$, we need to find $g \in \overline{\mathcal{A}}$ such that $|||f| - g||_\infty < \epsilon$.

   Set $M = ||f||_\infty$, so that $f(x) \in [-M, M]$. Since the function $x \mapsto |x|$ is continuous, by the classical Weierstrass theorem, we can find a polynomial $P : [-M, M] \to \mathbb{R}$ with real coefficients, namely

   $$P(t) = \sum_{k=0}^{n} a_k t^k \quad \text{such that} \quad \sup_{t \in [-M,M]} \big||t| - P(t)\big| < \epsilon. \tag{1}$$

   Now set $t = f(x)$, and $g(x) = P \circ f(x) = \sum_{k=0}^{n} a_k (f(x))^k$. Since $f \in \overline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ is an algebra, we have that $g \in \overline{\mathcal{A}}$. Further

   $$\big|||f| - g\big||_\infty = \sup_{x \in X}\big||f(x)| - P \circ f(x)\big| \leq \sup_{t \in [-M,M]} \big||t| - P(t)\big| < \epsilon,$$

   completing the proof. □

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3. Given an arbitrary metric space X, verify whether the following statement is true or false: $C(X;\mathbb{R})$ *always separates points and vanishes at no point.*

   Hint. Try the functions $x \mapsto d(x, x_0)$, for any fixed $x_0 \in X$. Use the triangle inequality to show that these functions are continuous. $\square$

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4. Find a compact metric space (X, d) and algebras $\mathcal{A}, \mathcal{B} \subseteq C(X;\mathbb{R})$ such that

   (a) $\mathcal{A}$ separates points but vanishes at some point.

   (b) $\mathcal{B}$ vanishes at no point but fails to separate points.

   Hint. Verify that these work.

$$\mathcal{A} = \{f \in C[0,1] : f(1/2) = 0\}, \qquad \mathcal{B} = \{f \in C[0,1] : f(0) = f(1)\}.$$

$\square$

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5. Evaluate with justification

$$\lim_{n\to\infty} \int_0^{\pi} \frac{n + \sin nx}{3n - \sin^2 nx}\, dx.$$

   Hint. Since

$$\left| \frac{n + \sin nx}{3n - \sin^2 nx} - \frac{1}{3} \right| = \left| \frac{3\sin nx + \sin^2 nx}{3(3n - \sin^2 nx)} \right| \le \frac{4}{3(3n-1)} \to 0 \text{ as } n \to \infty,$$

   we conclude that the integrand converges uniformly to the constant function $1/3$. Since uniform convergence permits interchange of limit and integration, the integral is $\pi/3$. $\square$

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6. For each $n \in \mathbb{N}$, you are given a differentiable function $f_n : \mathbb{R} \to \mathbb{R}$ that satisfies

$$f_n(0) = 2019, \qquad |f_n'(t)| \le 321 + |t|^{201} \text{ for all } t \in \mathbb{R}.$$

   Prove that there exists $f : \mathbb{R} \to \mathbb{R}$ and a subsequence f_{n_k} with the following property: for every compact subset K of $\mathbb{R}$, $f_{n_k} \to f$ uniformly on K. Clearly identify the principal theorems and methods that you apply.

   Hint. Prove that the family of functions $\{f_n : n \ge 1\}$ is equicontinous. Use Arzela-Ascoli to find susequences $S_1 \supset S_2 \supset \cdots \supset S_j \supset \cdots$ such that $\{f_n : n \in S_j\}$ is uniformly convergent on $[-j, j]$. Diagonalize. $\square$

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7. The Arzela-Ascoli theorem can be rephrased as: *"if $\{f_n : n \ge 1\}$ is a sequence of real-valued, equicontinuous and uniformly bounded functions on a compact metric space X, then $\{f_n\}$ has a uniformly convergent subsequence".* A website you found claims that the following is a proof of this theorem. Determine whether the proof is correct or incorrect.

   Proof. Let $\mathcal{F} = \{f_n : n \ge 1\}$. Since $\mathcal{F}$ is equicontinuous and X is compact, given $\epsilon > 0$, there is $\delta > 0$ such that

(2)
$$d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \text{ for all } n.$$

Compact also implies totally bounded, so X is totally bounded. Hence we can find finitely many points $\{x_k : 1 \le k \le K\} \subseteq X$ such that

$$X = \bigcup_{k=1}^{K} B(x_k; \delta).$$

Recall that $\mathcal{F}$ is known to be uniformly bounded (say by the finite constant M), so each one of the K sequences $\{f_n(x_k) : n \ge 1\}$ is an infinite sequence in $[-M, M]$. By the compactness of $[-M, M]$, and we can find subsequence $n_1 < n_2 < \cdots < n_\ell < \cdots \to \infty$ such that

$$\lim_{\ell \to \infty} f_{n_\ell}(x_k) \text{ exists for all } 1 \le k \le K.$$

Since any convergent sequence is Cauchy, we can find N such that for all $n_\ell, n_{\ell'} \ge N$,

$$|f_{n_\ell}(x_k) - f_{n_{\ell'}}(x_k)| < \frac{\epsilon}{3} \quad \text{for all } n_\ell, n_{\ell'} \ge N \text{ and for all } 1 \le k \le K. \tag{3}$$

We claim that $\{f_{n_\ell} : \ell \ge 1\}$ is the desired uniformly convergent subsequence. This will follow if we can show that $\{f_{n_\ell} : \ell \ge 1\}$ is uniformly Cauchy. To see this, given $x \in X$, first identify the index k such that $d(x, x_k) < \delta$. Then for all $n_\ell, n_{\ell'} \ge N$,

$$\begin{aligned} |f_{n_\ell}(x) - f_{n_{\ell'}}(x)| &\le |f_{n_\ell}(x) - f_{n_\ell}(x_k)| + |f_{n_\ell}(x_k) - f_{n_{\ell'}}(x_k)| + |f_{n_{\ell'}}(x_k) - f_{n_{\ell'}}(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

In the estimation above, we have used the equicontinuity criteron (2) to bound the first and the third term, and the pointwise Cauchy criterion (3) to bound the second. This allows us to conclude that $||f_{n_\ell} - f_{n_{\ell'}}||_\infty < \epsilon$ for all $n_\ell, n_{\ell'} \ge N$, and we are done. □

Hint. The subsequence n_ℓ depends on ϵ; it should be chosen at the outset and should work for all ϵ. □