Math 321, Spring 2019
Midterm 2, March 15

Name: **SID:**

Instructions

- The total time is 50 minutes.
- The total score is 80 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Partial credit will be assigned to the clarity and presentation style of solutions, so please ensure that your answers are effectively communicated.

Problem	**Points**	**Score**
1	30	
2	30	
3	20	
4 (Extra credit)	20	
MAX	80	

1. (a) When is a function $\alpha : [a,b] \to \mathbb{R}$ said to be of bounded variation?

(5 points)

(b) Determine whether the function $\alpha : [0,1] \to \mathbb{R}$ given by

$$\alpha(x) = \begin{cases} \log(1+x)\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is of bounded variation.

(10 points)

(c) A linear functional $L : C[0,1] \to \mathbb{R}$ obeys the following property: for every continuously differentiable $g : [0,1] \to \mathbb{R}$,

$$L(g) = -\int_0^1 g'(x)\cos(\pi x)\,dx.$$

Does there exist $\alpha \in \mathrm{BV}[0,1]$ such that

$$L(f) = \int_0^1 f(x)d\alpha(x), \text{ for every } f \in C[0,1]?$$

If yes, find such a function α. If not, explain why not. Clearly state any result you need to use.

(15 points)

2. For each of the following statements, determine whether it is true or false. The answer should be in the form of a short proof or an example, as appropriate.

   (a) There exists a bounded function on $[a, b]$ that fails to be Riemann-Stieltjes integrable with respect to every nondecreasing non-constant integrator α.

   (6 points)

   (b) The class $C[a, b]$ consists of all functions that are Riemann-Stieltjes integrable on $[a, b]$ with respect to every nondecreasing integrator α.

   (6 points)

(c) The Fourier series of a continuous 2π-periodic function f converges to f in the L^1 norm $||\cdot||_1$. Recall

$$||f||_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|\, dx.$$

(6 points)

(d) For any bounded, Riemann integrable function f, the sequence of Fourier coefficients $\{\widehat{f}(k) : k \geq 0\}$ converges to zero.

(6 points)

(e) Let f be a bounded Riemann integrable function on $[-\pi, \pi]$. Then $||\sigma_N f - f||_1 \to 0$ as $N \to \infty$. Here $\sigma_N f$ denotes the Nth partial Cesaro sum of f.

(6 points)

3. Let $\alpha, \beta > 0$. Evaluate the sum

$$\sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{2\alpha} y^{2\beta} \cos\big(m(x+y)\big)\, dy\, dx.$$

(20 points)

4. (Extra credit) Define the *frequency support* of a function f to be

$$\operatorname{supp}(\widehat{f}) := \left\{ n \in \mathbb{Z} : \widehat{f}(n) \neq 0 \right\},$$

where $\widehat{f}(n)$ denotes the n-th Fourier coefficient. Let $\mathcal{F}$ denote the class of all continuous functions whose frequency support is contained in $[-10, 10]$. Given any "gap" sequence $\{d_k : k \geq 1\} \subseteq \mathbb{N}$, find a continuous function g with the following frequency-replicating feature: for every $f \in \mathcal{F}$,

$$\operatorname{supp}\left[\widehat{(fg)}\right] = \bigcup_{k=1}^{\infty} A_k, \text{ with}$$

$$A_k := \{a_k + n : n \in \operatorname{supp}(\widehat{f})\} \text{ for some integer } a_k, \text{ and}$$

$$\operatorname{dist}(A_k, A_{k'}) \geq d_k + \cdots + d_{k'-1} \text{ for all } k < k'.$$

(20 points)