Math 321 Assignment 9
Due Wednesday, March 13 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. In HW 8 problem 2, you showed that the Fourier series of a twice continuously differentiable 2π-periodic function converges absolutely and uniformly to the function itself. Let us now try to generalize this result by weakening the smoothness hypotheses on the function a bit more. Show that if $f \in \mathcal{C}^{2\pi}$ is once continuously differentiable, then $S_n f \to f$ uniformly as $n \to \infty$, where $S_n f$ denotes the nth partial Fourier sum. *Remark: This result continues to hold even if f is Hölder-continuous of order $\alpha \in (0, 1]$, but you do not need to prove it here.*

2. In class, we sketched a proof of Fejér's theorem: *if $f \in \mathcal{C}^{2\pi}$, then $\sigma_n(f)$ converges uniformly to f as $n \to \infty$*. Fill in the details of the proof. Use Fejér's theorem to give yet another proof of Weierstrass's second theorem.

3. Let $f \in \mathcal{C}^{2\pi}$ = the class of continuous 2π-periodic functions on $[-\pi, \pi]$. We have seen that the partial Fourier sums $s_n(f)$ are excellent approximations of f in L^2 norm, but so far only have indirect evidence that they may not be very good approximations in the sup norm. This problem attempts to make this intuition precise.

   (a) Let us start with a small step, namely by finding a bounded function with at least one large partial Fourier sum. Show that for every $n \geq 1$, there exists $f_n \in \mathcal{C}^{2\pi}$ such that $||f_n||_\infty = 1$ and $\sup_j |s_j f_n(0)| > n$.

   (b) The functions you found serve as building blocks for the counterexample we seek. Use the functions f_n in part (a) to find a single $f \in \mathcal{C}^{2\pi}$ whose Fourier series diverges at 0.

   (c) Now modify your construction in part (b) to create a continuous 2π-periodic function whose Fourier series diverges on a dense set of points.

   (d) Why does the divergence of the Fourier series of a continuous function not contradict Weierstrass's second approximation theorem?

4. This exercise is designed to study a curious property of a certain class of Fourier series, known as Gibbs phenomenon. Discovered by Wilbraham (1849) and studied by Gibbs (1899), this phenomenon refers to the manner in which the Fourier series of a piecewise continuously differentiable periodic function behaves at a jump discontinuity. The nth partial Fourier sums oscillate near the jump point, which is understandable, but the strange thing is that the oscillation might result in increasing the maximum of the partial sum above that of the function itself. Even more strange is the fact that the overshoot does not die out as you take larger and larger sums (i.e. the frequency increases), but approaches a finite nonzero limit! Here is an example where you can see Gibbs phenomenon in action.

   Let f denote the 2π-periodic function given by $f(x) = \text{sgn}(x)$ on $(-\pi, \pi]$, which takes on the value 1, -1 or 0 according as x is positive, negative or zero.

(a) Show that the formal Fourier series of f is given by

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{(2n-1)}.$$

This series in known to converge to $f(x)$ at all points $x \in (-\pi, \pi)$, but you do not need to prove this here.

(b) Denote by s_n the nth partial sum of the above series. Show that

$$s_n(x) = \frac{2}{\pi}\int_0^x \frac{\sin 2nt}{\sin t}\,dt.$$

(c) Examine the local maxima and minima of s_n, and deduce that the largest value of s_n is attained at $\frac{\pi}{2n}$.

(d) Interpret $s_n(\frac{\pi}{2n})$ as a Riemann sum and prove that

$$\lim_{n\to\infty} s_n\left(\frac{\pi}{2n}\right) = \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt.$$

The value of this limit is about 1.179. Thus, although f has a jump equal to 2 at the origin, and although the Fourier series of f converges to f at the origin, the graphs of the approximating curves s_n tend to approximate a vertical segment of length 2.358 in the vicinity of the origin!