Math 321 Assignment 7
Due Wednesday, February 27 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. Helly's first and second theorems, stated below, are convergence results on the space $\mathrm{BV}[a, b]$. They are critical components in our proof of the Riesz representation theorem for continuous linear functionals on $C[a, b]$. You proved the first one in HW 6. Now prove the second.

   (a) (*Helly's first theorem*) Let $\{f_n\}$ be a bounded sequence in $\mathrm{BV}[a, b]$, i.e., suppose that $||f_n||_{\mathrm{BV}} \leq K$ for all n. Show that f_n admits a pointwise convergent subsequence whose limit f lies in $\mathrm{BV}[a, b]$ with $||f||_{\mathrm{BV}} \leq K$.

   (b) (*Helly's second theorem*) Suppose that $\{\alpha_n\}$ is a sequence in $\mathrm{BV}[a, b]$. If $\alpha_n \to \alpha$ pointwise on $[a, b]$, and if $V_a^b \alpha_n \leq K$ for all n, then $\alpha \in \mathrm{BV}[a, b]$ and

$$\int_a^b f \, d\alpha_n \to \int_a^b f \, d\alpha \quad \text{for all } f \in C[a, b].$$

2. This problem addresses the issue of uniqueness of the integrator α in Riesz representation theorem. Fill in the following steps.

   (a) Given $\alpha \in \mathrm{BV}[a, b]$, define $\beta(a) = \alpha(a)$, $\beta(x) = \alpha(x+)$ for $a < x < b$ and $\beta(b) = \alpha(b)$. Show that β is right continuous on (a, b), that $\beta \in \mathrm{BV}[a, b]$ and that

$$\int_a^b f \, d\alpha = \int_a^b f \, d\beta \quad \text{for every } f \in C[a, b]. \tag{1}$$

   (b) Given $\alpha \in \mathrm{BV}[a, b]$, show that there is a unique $\beta \in \mathrm{BV}[a, b]$ with $\beta(a) = 0$ such that β is right continuous on (a, b) and (1) holds.

   (c) Given a continuous linear functional $L : C[a, b] \to \mathbb{R}$, we constructed a function $\alpha \in \mathrm{BV}[a, b]$ such that

$$L(f) = \int_a^b f \, d\alpha.$$

   Argue that such α is not unique in general. However, combine the above steps to conclude that α can be chosen to be right continuous on (a, b) with $\alpha(a) = 0$, and that in this case, α is unique.

3. For $f \in C[0, 1]$, does the following limit exist? If yes, evaluate it. If not, explain why not.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=2}^{n} f\left(\frac{\log k}{\log n}\right).$$

4. Many of the results we have studied in this course thus far focus on approximation of continuous functions by other special functions, such as polynomials or trigonometric sums or polygonal functions, usually in the sup norm. Can one approximate Riemann-integrable functions, which are not necessarily continuous, by continuous functions and in some suitable norm? This problem addresses one aspect of this issue of approximation.

   Suppose that f is a bounded, 2π-periodic function that is Riemann-integrable on $[-\pi, \pi]$.

   (a) Show that there is a continuous function g on $[-\pi, \pi]$ satisfying $||f - g||_2 < \epsilon$. Recall that $|| \cdot ||_2$ is the norm given by

   $$||f||_2 := \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx \right]^{\frac{1}{2}}.$$

   (b) Can the function g above be chosen to be continuous and 2π-periodic?

   (c) Can g be chosen to be a trigonometric polynomial?