Math 321 Assignment 6
Due Wednesday, February 20 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. Recall Jordan's theorem: a function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if f can be written as the difference of two non-decreasing functions g and h.

   (a) Show that the decomposition $f = g - h$ is by no means unique, and that there are uncountably many ways of writing f in this form.

   (b) The following decomposition of f is often useful. Define the positive and negative variations of f by

   $$p(x) = \frac{1}{2}\big(v(x) + f(x) - f(a)\big), \qquad n(x) = \frac{1}{2}\big(v(x) - f(x) + f(a)\big),$$

   where $v(x) = V_a^x f$ is the variation function defined in class. Show that p and n are nondecreasing functions on $[a, b]$ and use this to give an alternative representation of f as the difference of nondecreasing functions.

   (c) The relevance of p and n is that it injects a certain amount of uniqueness into the Jordan decomposition of f, in the following sense. If g and h are any two non-decreasing functions on $[a, b]$ such that $f = g - h$, then

   $$V_x^y p \le V_x^y g \quad \text{and} \quad V_x^y n \le V_x^y h \text{ for all } x < y \text{ in } [a, b].$$

   Prove this.

2. We stated the "integration by parts" formula in class, using it to highlight the interchangability of integrand and integrator. The purpose of this problem is to fill in the details of its proof. Throughout this problem f and α denote arbitrary real-valued functions on $[a, b]$.

   (a) Given any partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ and a collection of points $T = \{t_1, \cdots, t_n\}$ with $t_j \in [x_{j-1}, x_j]$, prove the following identity:

   $$S_f(\alpha, P, T) = f(b)\alpha(b) - f(a)\alpha(a) - S_\alpha(f, P', T').$$

   Here $P' = \{a = t_0, t_1, \cdots, t_n, t_{n+1} = b\}$ and $T' = P$. Recall that $S_\alpha(f, P, T)$ is our notation for a Riemann-Stieltjes sum corresponding to the integrand f, integrator α, partition P and selection of points T.

   (b) Use part (a) to show that $f \in \mathcal{R}_\alpha[a, b]$ if and only if $\alpha \in \mathcal{R}_f[a, b]$. Show that in either case,

   $$\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a).$$

   Note that this is one of those rare instances where one implication implies the other!

(c) Use the integration by parts formula to prove the following statement. If $\alpha \in \mathrm{BV}[a, b]$, then $C[a, b] \subseteq \mathcal{R}_\alpha[a, b]$. Recall that we proved this result in class for nondecreasing integrators α. The statement here extends this to integrators of bounded variation.

3. In HW 5 Problem 1, we saw that if α is a nondecreasing integrator, then the space $\mathcal{R}_\alpha[a, b]$ is rich in algebraic and analytical properties; in particular, it is a vector space, an algebra and a lattice. In this problem we show that these desirable properties carry over for integrators α of bounded variation.

   Let $\alpha \in \mathrm{BV}[a, b]$ and let $\beta(x) = V_a^x \alpha$. Recall that both β and $\beta - \alpha$ are increasing.

   (a) Show that $\mathcal{R}_\alpha[a, b] = \mathcal{R}_\beta[a, b] \cap \mathcal{R}_{\beta-\alpha}[a, b]$. (*Hint:* Argue that it suffices to only show that $\mathcal{R}_\alpha[a, b] \subseteq \mathcal{R}_\beta[a, b]$.)

   (b) Use HW 5 Problem 1 and the identity above to conclude that $\mathcal{R}_\alpha[a, b]$ is a vector space, an algebra and a lattice.

   (c) Suppose now that β is any nondecreasing function (not necessarily $V_a^x \alpha$) such that $\beta - \alpha$ is also nondecreasing. Is the identity in part (a) necessarily true in this more general context? Prove or give a counterexample.

4. Let $\{f_n\}$ be a bounded sequence in $\mathrm{BV}[a, b]$, i.e., suppose that $||f_n||_{\mathrm{BV}} \leq K$ for all n. Show that f_n admits a pointwise convergent subsequence whose limit f lies in $\mathrm{BV}[a, b]$ with $||f||_{\mathrm{BV}} \leq K$. This is known as Helly's first theorem. (*Hint:* First try out the case when all the functions f_n are non-decreasing, then adapt it for functions of bounded variation.)