Math 321 Assignment 11
Due Wednesday, April 3 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by

$$f_1(x,y) = e^x \cos y, \qquad f_2(x,y) = e^x \sin y.$$

   (a) What is the range of $\mathbf{f}$?

   (b) Show that $\mathbf{f}$ is locally injective at every point in $\mathbb{R}^2$, but not globally injective on $\mathbb{R}^2$.

   (c) The inverse function theorem ensures that $\mathbf{f}$ has a continuous inverse $\mathbf{g}$ near every point $\mathbf{a}$. Find an explicit formula for $\mathbf{g}$ with $\mathbf{a} = (0, \pi/3)$. Compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, where $\mathbf{b} = \mathbf{f}(\mathbf{a})$.

2. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

   can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

3. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

   Show that there exists a differentiable function g in some neighbourhood of $(1, -1)$ in $\mathbb{R}^2$ such that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$. Find the first partial derivatives of g at $(1, -1)$.

4. The contraction mapping principle, explicitly stated in Banach's thesis in 1922, has featured prominently in our discussions this past week, as a tool in the proof of the inverse function theorem. This result has a number of interesting implications beyond the ones we have seen in class, some of which we will explore in this problem set.

   (a) Let us start by proving the principle itself: Let (M, d) be a complete metric space, and let $f : M \to M$ be a strict contraction, i.e., there exists a constant $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \text{ for all } x, y \in M, x \neq y.$$

   Show that f has a unique fixed point. Moreover, given any point $x_0 \in M$, the sequence of functional iterates $\{f^n(x_0) : n \geq 1\}$ (also called the orbit of x_0 under f) always converges to the fixed point for f. Here f^n denotes the composition of f with itself n times.

(b) Is the converse of the contraction mapping principle true, namely, does the existence of a fixed point force the function to be a contraction, at least locally? The answer is no, as the following example illustrates.

The function $f(x) = x^2$ has two obvious fixed points: $p_0 = 0$ and $p_1 = 1$. Show that there is a $0 < \delta < 1$ such that

$$|f(x) - p_0| < |x - p_0| \text{ whenever } |x - p_0| < \delta,\ x \neq p_0.$$

Conclude that $f^n(x) \to p_0$ for all such x. We say that p_0 is an *attracting fixed point* for f; every orbit that starts out near p_0 ends at p_0.

In contrast, find a $\delta > 0$ such that

$$|f(x) - p_1| > |x - p_1| \text{ whenever } |x - p_1| < \delta, x \neq p_1.$$

This means that p_1 is a *repelling fixed point* for f; orbits that start out near p_1 are pushed away from p_1. In fact, show that $f^n(x) \not\to 1$ as $n \to \infty$, for any real x with $|x| \neq 1$.

(c) The intuition gathered from the preceding example can be used to prove the more general result. Suppose that $f : (a, b) \to (a, b)$ has a fixed point p in (a, b) and that f is differentiable at p. If $|f'(p)| < 1$, prove that p is an attracting fixed point for f. If $|f'(p)| > 1$, prove that p is a repelling fixed point for f.

(d) The findings in (c) lead to the natural question: can we say anything about the nature of a fixed point p if $|f'(p)| = 1$? Apparently not, as the following examples show:

$$f_1(x) = \arctan x,\ p_1 = 0, \qquad f_2(x) = x^3 + x,\ p_2 = 0 \qquad f_3(x) = x^2 + \frac{1}{4},\ p_3 = \frac{1}{2}.$$

Show that for each $j = 1, 2, 3$, the point p_j is a fixed point for f_j, with $f_j'(p_j) = 1$. Then show that p_1 is an attracting fixed point, p_2 is repelling and p_3 is neither.

(e) Now put your knowledge to the test. First justify that the cubic $x^3 - x - 1$ has a unique real root $x_0 \in (1, 2)$. Our goal is to devise an iterative algorithm for finding this root numerically. Find a function f, on which the contraction mapping principle can be applied, so that for any $x \in [1, 2]$, the sequence of approximations $x_n = f^n(x) \to x_0$ as $n \to \infty$. (*Hint: The obvious choice of f does not work, so proceed with caution.*)