Math 321 Assignment 1
Due Wednesday, January 9 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) Please staple your pages together when you submit your assignment.

(iv) Do not forget to include your name and SID.

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1. Let $\{f_n : n \geq 1\}$ be a sequence of functions, with $f_n : [a,b] \to \mathbb{R}$ and $|f_n(x)| \leq 1$ for all $x \in [a,b]$ and all $n \geq 1$. Show that there is a subsequence $\{f_{n_k} : k \geq 1\}$ such that

$$\lim_{k\to\infty} f_{n_k}(x) \text{ exists for each rational number } x \in [a,b].$$

2. For each $n \geq 1$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a function such that the sequence $f_n \to 0$ uniformly on every compact interval of $\mathbb{R}$. Does it follow that f_n converges uniformly to zero on $\mathbb{R}$? Explain.

3. For each of the following sequences, determine the pointwise limit on the given interval (if it exists) and the intervals on which the convergence is uniform (if any):

   (a) $f_n(x) = n^2 x(1-x^2)^n$ on $[0,1]$,

   (b) $f_n(x) = xe^{-nx}$ on $[0,\infty)$.

4. Let $C(\mathbb{R})$ denote the space of all continuous functions on $\mathbb{R}$. Define a metric d on $C(\mathbb{R})$ by setting

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(f,g)}{1+d_n(f,g)}, \qquad \text{where} \qquad d_n(f,g) = \max_{|t|\leq n} |f(t)-g(t)|.$$

   You do not need to submit a proof verifying that d is a metric.

   (a) Given $f, f_n \in C(\mathbb{R})$, prove that $d(f_n, f) \to 0$ if and only if f_n converges uniformly to f on every compact subset of $\mathbb{R}$. For this reason, convergence in $C(\mathbb{R})$ is sometimes called *uniform convergence on compacta.*

   (b) Show that $C(\mathbb{R})$ is complete.

5. If $\sum_n |a_n| < \infty$, prove that the series

$$\sum_{n=1}^{\infty} a_n \sin nx \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \cos nx$$

   are uniformly convergent on $\mathbb{R}$.