Math 421/510 Quiz 6 Solution

1. Find a necessary and sufficient condition on a sequence $\{\alpha_n : n \geq 1\}$ so that the linear operator A given by

$$A\mathbf{e}_n = \alpha_n \mathbf{e}_n \tag{1}$$

is bounded on $\ell^2(\mathbb{N})$. Here $\{e_n : n \geq 1\}$ is the canonical orthonormal basis in $\ell^2(\mathbb{N})$.

(10 points)

Proof. Our claim is that an operator A satisfying (1) admits a continuous extension to $\ell^2(\mathbb{N})$ if and only if $\{\alpha_n\}$ is a bounded sequence.

To establish necessity, suppose that A is continuous. Then

$$|\alpha_n| = ||A\mathbf{e}_n||_2 \leq \sup_{||\mathbf{x}||=1} ||A\mathbf{x}|| = ||A||_{\text{op}},$$

proving that the sequence $\{\alpha_n\}$ is bounded.

To prove sufficiency, suppose that $\{\alpha_n\}$ is a sequence bounded by M. Let $\mathcal{E}$ denote the span of $\{e_n : n \geq 1\}$, i.e., the space of finite linear combinations of the orthonormal basis vectors. The assumption (1) combined with linearity of A implies that

$$||A\mathbf{x}|| = ||\sum_{n=1}^{N} \alpha_n x_n \mathbf{e}_n|| = \big(\sum_{n=1}^{N} |\alpha_n x_n|^2\big)^{\frac{1}{2}} \leq M||x|| \quad \text{for all } x = \sum_{n=1}^{N} x_n \mathbf{e}_n \in \mathcal{E}. \tag{2}$$

Since $\mathcal{E}$ is dense in the Banach space $\ell^2(\mathbb{N})$, the above inequality yields a unique continuous extension of A to $\ell^2(\mathbb{N})$ via the following prescription: if $\mathbf{u} \in \ell^2(\mathbb{N})$, let $\mathbf{u}_k \in \mathcal{E}$ be a sequence such that $\mathbf{u}_k \to \mathbf{u}$. By (2),

$$||A(\mathbf{u}_k - \mathbf{u}_j)|| \leq M||\mathbf{u}_k - \mathbf{u}_j|| \to 0 \text{ as } j, k \to \infty.$$

Thus $\{A\mathbf{u}_k\}$ is a Cauchy sequence in $\ell^2(\mathbb{N})$, hence convergent. We set

$$A\mathbf{u} := \lim_{k\to\infty} A\mathbf{u}_k.$$

It is easy to check from (2) that the definition is independent of the choice of the sequence $\{\mathbf{u}_k\}$ and defines a bounded linear operator on all of $\ell^2(\mathbb{N})$. □