Math 421/510 Homework 5, Spring 2018

1. Given any point $t_0 \in [0, 2\pi]$, show using the uniform boundedness principle that there exists a continuous 2π-periodic function whose Fourier series diverges at t_0. We sketched a proof of this result in class. Fill in the details.

2. In class, we introduced the concept of a locally convex space, whose topology is generated by a family of seminorms. When is such a topology equivalent to a metric topology? A norm topology?

3. Let (X, Ω, μ) be a σ-finite measure space, $1 \le p < \infty$. Suppose that $K : X \times X \to \mathbb{F}$ is an $\Omega \times \Omega$-measurable function such that for $f \in L^p(\mu)$ and almost every $x \in X$, the function $K(x, \cdot) f(\cdot) \in L^1(\mu)$ and
$$\mathcal{K}f(x) = \int K(x,y) f(y)\, d\mu(y)$$
defines an element $\mathcal{K}f \in L^p(\mu)$. Show that $\mathcal{K}$ is a bounded operator on $L^p(\mu)$.

4. (a) Show that the weak topology on a locally convex space X is the smallest topology such that each $\ell \in X^*$ is continuous.

   (b) Show that the weak-star topology is the smallest topology such that for each $x \in X$, the map $\ell \mapsto \ell(x)$ is continuous.

5. (a) If $\mathbb{H}$ is a Hilbert space and $\{h_n\} \subset \mathbb{H}$ is a sequence such that $h_n \to h$ weakly and $||h_n|| \to ||h||$, then show that $h_n \to h$ strongly.

   (b) Prove the same statement for the Lebesgue spaces $L^p(\mu)$, $1 < p < \infty$.

6. Suppose that X is an infinite-dimensional normed space. Find the weak closure of the unit sphere.