Math 421/510 Homework 2, Spring 2018

1. Let X be an infinite-dimensional Banach space. Show that every Hamel basis of X is uncountable.

2. (a) Show that the vector space of polynomials is dense in $C[0,1]$ equipped with the sup norm, but the monomials $\{x^n : n \geq 0\}$ do not form a Schauder basis of $C[0,1]$.

   (b) Does $C[0,1]$ have a Schauder basis? If yes, find one. If not, explain why not.

3. (a) Let X be a normed space, and Y a proper subspace. Denote by X^* the space of all bounded linear functionals on X. Show that if $\ell \in Y^*$, then there exists $L \in X^*$ such that $L|_Y \equiv \ell$ and $||L|| = ||\ell||$.

   (b) Use the above to show that if X is a normed vector space and $x \in X$, then
$$||x|| = \sup\{|\ell(x)| : \ell \in X^* \text{ and } ||\ell|| \leq 1\}.$$

4. Here is another "separation" theorem for you to prove: Let Y be a proper closed subspace of X, $u \in X \setminus Y$ and $\rho = \text{dist}(u, Y)$. Show that there exists a linear functional $\ell \in X^*$ such that $\ell(u) = 1$, $\ell \equiv 0$ on Y, and $||\ell|| = \rho^{-1}$.

5. Show that there is a linear functional ℓ of norm 1 on the space of real bounded sequences that generalizes the concept of limits, in the following sense:
   - ℓ is shift-invariant, i.e. $\ell(x_1, x_2, \cdots) = \ell(x_2, x_3, \cdots)$,
   - $\ell(x) = \lim_{n\to\infty} x_n$ for convergent sequences $x = (x_1, x_2, \cdots)$,
   - ℓ is nonnegative for nonnegative sequences.

   A linear functional of this type is called a *Banach limit*.