MATH 421/510
Assignment 1

Suggested Solutions

January 2018

All exercises from Folland, section 5.1. "Problem 1 (6)" below, for instance, indicates that the first problem on this assignment is Exercise 5.1.6 in Folland.

Problem 1 (6). *Suppose that X is a finite-dimensional normed space. Let $e_1, e_2, \dots, e_n$ be a basis for X, and define $\|\sum_1^n a_j e_j\|_1 = \sum_1^n |a_j|$.*

a) $\|\cdot\|_1$ is a norm on X.

b) The map $(a_1, \dots, a_n) \mapsto \sum_1^n a_j e_j$ is continuous from K^n with the usual Euclidean topology to X with the topology defined by $\|\cdot\|_1$.

c) $\{x \in X : \|x\|_1 = 1\}$ is compact in X.

d) All norms on X are equivalent. (Compare any norm to $\|\cdot\|_1$.)

Solution.

a) As always, we need to check positivity, homogeneity, and subadditivity.

- **Positivity**:
  Let $x = \sum_{j=1}^n a_j e_j$. Then $\|x\|_1 \geq 0$.
  Suppose that $x \neq 0$. Then $a_j \neq 0$ for some $1 \leq j \leq n$, and hence

$$\|x\|_1 = \sum_{i=1}^n |a_i| \geq |a_j| > 0$$

  Conversely, suppose that $x = 0$. Then $a_j = 0$ for each $1 \leq j \leq n$, so $\|x\|_1 = 0$.
- **Homogeneity**: Let $x = \sum_{j=1}^n a_j e_j$ as before and let $c \in K$. We compute directly:

$$\|cx\|_1 = \left\| c\sum_{j=1}^n a_j e_j \right\|_1 = \left\| \sum_{j=1}^n (ca_j) e_j \right\|_1 = \sum_{j=1}^n |ca_j| = |c| \sum_{j=1}^n |a_j| = |c| \|x\|_1$$

  as required.

- **Subadditivity**: If $x = \sum_{j=1}^n a_j e_j$ and $b = \sum_{j=1}^n b_j e_j$, then

$$\|x+y\|_1 = \left\| \sum_{j=1}^n (a_j + b_j) e_j \right\|_1 = \sum_{j=1}^n |a_j + b_j| \leq \sum_{j=1}^n |a_j| + \sum_{j=1}^n |b_j| = \|x\|_1 + \|y\|_1.$$

b) Let $\|\cdot\|_2$ denote the Euclidean norm on K^n.

Since the map $\phi(a_1, \ldots, a_n) := \sum_1^n a_j e_j$ is linear, it suffices to show that the map is continuous at the origin. Let $0 < \varepsilon < 1$, and let $\delta = \varepsilon/\sqrt{n}$. Let $\vec{a} = (a_1, \ldots, a_n)$ and suppose that $\|\vec{a}\|_2 < \delta$. Then by the Cauchy-Schwarz inequality,

$$\sum_{j=1}^n |a_j| \leq \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^n 1^2 \right)^{\frac{1}{2}} < \delta\sqrt{n} = \varepsilon.$$

This finishes to proof.

c) Consider the continuous mapping $\phi : (a_1, \ldots, a_n) \to \sum_1^n a_j e_j$ defined in (b). If we denote $S := \{x \in X : \|x\|_1 = 1\}$ and the "polyhedron"

$$P = \{(a_1, a_2, \ldots, a_n) \in K^n : \sum_{j=1}^n |a_j| = 1\},$$

which is compact by the Heine-Borel Theorem, we see that $S = \phi(P)$ is compact.

d) Given any norm $\|\cdot\|$ on X, it suffices to show that there are $c > 0, C < \infty$ such that for all $x \in S$ (recall $S := \{x \in X : \|x\|_1 = 1\}$), we have

$$c \leq \|x\| \leq C.$$

Write $x = \sum_{j=1}^n a_j e_j$. Then

$$\|x\| = \left\| \sum_{j=1}^n a_j e_j \right\| \leq \sum_{j=1}^n |a_j| \|e_j\| \leq C \sum_{j=1}^n |a_j| = C\|x\|_1 = C.$$

Here, $C := \max_{j=1}^n \|e_j\| < \infty$.

To prove the lower bound, let $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|)$ be the identity map. The above shows that I is a bounded linear map, and hence continuous. Since taking norm is continuous, the map $\sigma : (X, \|\cdot\|_1) \to \mathbb{R}$ by $\sigma(x) := \|I(x)\|$ is continuous.

But S is compact and σ is continuous, hence $\sigma(S)$ is compact in $\mathbb{R}$, and thus has a minimum. Since $I(S) = S$ does not contain the origin, we have $0 \notin \sigma(S)$, which means that $c := \min \sigma(S)$ is strictly positive. This shows the lower bound.

Problem 2 (8)**.** *Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of complex measures on $(X, \mathcal{M})$. Then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space. (Use Theorem 5.1., which states that a normed vector space X is complete if and only if every absolutely convergent series in X converges. Also, $|\mu|$ is the total variation of the measure μ.)*

Solution. The properties of a norm are easily satisfied (consult Rudin if you have any questions).

- **Positivity**: The total variation $\|\mu\|$ of a measure μ is always nonnegative, and $|\mu|(X) = 0$ if and only μ is the zero measure.
- **Homogeneity**: This is direct.
- **Subadditivity**: This follows from the triangle inequality for complex numbers as well as for suprema.

To show that $M(X)$ is complete, we use Theorem 5.1.

Let ν_n be a sequence of complex measures on X such that $\sum_{n=1}^{\infty} \|\nu_n\| < \infty$. If we define $\nu(A) = \sum_{n=1}^{\infty} \nu_n(A)$ for every $A \in \mathcal{M}$ and show that ν is indeed a complex measure to which the series $\sum_{n=1}^{\infty} \nu_n$ converges in $M(X)$, then we are done.

Let $A \in M(X)$, then the series defining A converges absolutely since $|\nu_n(A)| \leq \|\nu_n\|$. ν is a complex measure: let $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ be disjoint. Then

$$\begin{aligned} \sum_{i=1}^{\infty} \nu(A_i) &= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \nu_n(A_i) \\ \text{(by Fubini's theorem)} &= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \nu_n(A_i) \\ &= \sum_{n=1}^{\infty} \nu_n(A) \\ &= \nu(A). \end{aligned}$$

Lastly, we show $\sum_{n=1}^{\infty} \nu_n$ converges to ν in $M(X)$.

Let $\{A_i\}_{i=1}^{\infty}$ be a measurable partition of X. Then

$$\begin{aligned} \sum_{i=1}^{\infty} \left| \sum_{n=1}^{N} \nu_n(A_i) - \nu(A_i) \right| &= \sum_{i=1}^{\infty} \left| \sum_{n=N+1}^{\infty} \nu_n(A_i) \right| \\ &\leq \sum_{i=1}^{\infty} \sum_{n=N+1}^{\infty} |\nu_n(A_i)| \\ \text{(by Fubini's theorem)} &= \sum_{n=N+1}^{\infty} \sum_{i=1}^{\infty} |\nu_n(A_i)| \\ \text{(by definition of total variation)} &\leq \sum_{n=N+1}^{\infty} \|\nu_n\|. \end{aligned}$$

Taking supremum with respect to the partition $\{A_i\}$, we have

$$\left\| \sum_{n=1}^{N} \nu_n - \nu \right\| \leq \sum_{n=N+1}^{\infty} \|\nu_n\|.$$

However, since the last series is absolutely convergent, letting $N \to \infty$, we are done.

Problem 3 (9)**.** *Let $C^k([0,1])$ be the space of functions on $[0,1]$ possessing continuous derivatives of order up to and including k, including one-sided derivatives at the endpoints.*

a) If $f \in C([0,1])$, then $f \in C^k([0,1])$ iff f is k times continuously differentiable on $(0,1)$ and $\lim_{x\searrow 0} f^{(j)}(x)$ and $\lim_{x\nearrow 1} f^{(j)}(x)$ exist for $j \leq k$. (The mean value theorem is useful.)

b) $\|f\| = \sum_0^k \|f^{(j)}\|_u$ is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach space. (Use induction on k. The essential point is that if $\{f_n\} \subset C^1([0,1])$, $f_n \to f$ uniformly, and $f_n' \to g$ uniformly, then $f \in C^1([0,1])$ and $f' = g$. The easy way to prove this is to show that $f(x) - f(0) = \int_0^x g(t)\,dt$.)

Solution.

a) Assume that f is real for the moment.

For the case $k = 1$, we need to show that for $f \in C([0,1])$ with a continuous derivative on $(0,1)$, the limit

$$f'(0) := \lim_{x\searrow 0} \frac{f(x) - f(0)}{x}$$

exists and is equal to $l := \lim_{x\searrow 0} f'(x)$; the result for higher k follows by induction.

Since $l = \lim_{x\searrow 0} f'(x)$ exists, for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f'(x) - l| < \varepsilon$ for all $0 < x < \delta$.

Let $0 < x < \delta$. Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, by the mean value theorem, there is $c \in (0,x)$ such that

$$f(x) - f(0) = xf'(c),$$

whence

$$\left|\frac{f(x) - f(0)}{x} - l\right| = |f'(c) - l| < \varepsilon.$$

This shows that $f \in C^1([0,1])$.

A very similar argument establishes the same result for $f'(1)$; in fact, we can just apply the same argument to $g(x) = f(1-x)$. Furthermore, suppose the result is true for all $j \leq k$, and choose $f \in C^k([0,1])$ such that f is continuously differentiable to order $k+1$ on $(0,1)$ and such that $\lim_{x\searrow 0} f^{(k+1)}(x)$ and $\lim_{x\nearrow 1} f^{(k+1)}(x)$ both exist. Then we can apply the above argument to $f^{(k)}$ to conclude that $f^{(k)} \in C^1([0,1])$, and hence that $f \in C^{k+1}([0,1])$. This completes the proof of sufficiency, while necessity is indeed part of the definition of C^k.

Lastly, if f is complex valued, then we split $f = \mathrm{Re}(f) + i\mathrm{Im}(f)$ and apply the above arguments to $\mathrm{Re}(f), \mathrm{Im}(f)$ respectively to get the result.

Remark: The mean value theorem fails for complex valued functions. Consider the example $f(x) := e^{ix}$ defined on $[0, 2\pi]$. We have $f(0) = f(2\pi) = 1$, but there is no $c \in (0, 2\pi)$ with $2\pi f'(c) = 0$.

b) The properties of a norm are easily satisfied.

- **Positivity**: The zero function has all its derivatives identically zero; conversely, any continuous function f on $[0,1]$ that is not identically zero has $\|f\| \geq \|f\|_u > 0$.
- **Homogeneity**: This follows from the homogeneity of the derivative, which follows from that of limits.
- **Subadditivity**: Every term in the sum defining the C^k norm is subadditive, so the sum must be as well.

The difficulty is completeness, which will finish off the requirements for $C^k([0,1])$ to qualify as a Banach space.

The proof, as suggested, proceeds by induction. The case $k = 0$ is well known, i.e. $C([0,1])$ is complete. Assume that the result is true for $C^k([0,1])$ for some $k \geq 0$, and choose a Cauchy sequence $(f_n) \subset C^{k+1}([0,1])$. This means in particular that $(f_n)_{n\geq 1}$ is a Cauchy sequence in $C^k([0,1])$. Since $C^k([0,1])$ is complete by the inductive hypothesis, there exists some $f \in C^k([0,1])$ with $f_n \to f$ in the topology of $C^k([0,1])$. In particular, this means that $f_n^{(k)} \to f^{(k)}$ uniformly. We also know that the sequence of continuous functions $(f_n^{(k+1)})_{n\geq 1}$ is uniformly Cauchy, which implies that there exists some function $g \in C([0,1])$ with $f_n^{(k+1)} \to g$ uniformly.

We can thus apply the C^1 result in the hint to the sequence $(f_n^{(k)})_{n\geq 1}$ to get $f^{(k+1)} = g$. Thus

$$\|f_n - f\|_{C^{k+1}} = \|f_n - f\|_{C^k} + \left\|f_n^{(k+1)} - f^{(k+1)}\right\|_u \to 0.$$

A great deal is now riding on the proof for C^1. Consider a Cauchy sequence $(f_n) \subset C^1([0,1])$. We know that there exist continuous functions f, g such that $f_n \to f$ and $f_n' \to g$ uniformly. Furthermore, by the triangle inequality and the fundamental theorem of calculus, we have

$$f_n(x) - f_n(0) = \int_0^x f_n'(t)dt.$$

Letting $n \to \infty$ and by uniform convergence of $f_n' \to g$, we have

$$f(x) - f(0) = \int_0^x g(t)dt.$$

Using the fundamental theorem of calculus again, we have for all x,

$$f'(x) = g(x).$$

The desired result, that C^k, $k \geq 0$ is a Banach space with respect to the given norm, has been proved.

Problem 4 (13). *If $\|\cdot\|$ is a seminorm on the vector space X, let $M = \{x \in X : \|x\| = 0\}$. Then M is a subspace, and the map $x + M \mapsto \|x\|$ is a norm on X/M.*

Solution. Recall that a if X is a linear space, then we say that a function $\|\cdot\| : X \to [0, \infty)$ is a seminorm on X if it satisfies the homogeneity and subadditivity requirements for a norm, and sends $0 \in X$ to $0 \in \mathbb{R}$, but might also vanish elsewhere in X.

First we need to show that M is a subspace. Indeed, $0 \in M$, and if $c \in K$, $x, y \in M$, then $\|cx + y\| \le |c|\|x\| + \|y\| = 0$, so $cx + y \in M$.

Then we need to show that the map $x+M \mapsto \|x\|$ is well-defined, that is, that if $x-y \in M$, then $\|x\| = \|y\|$. But $x = y + (x - y)$, and

$$\begin{aligned}\|x\| &= |y + x - y| \\ &\le \|y\| + \|(x - y)\| \\ &= \|y\|.\end{aligned}$$

By symmetry, we also have $\|x\| \ge \|y\|$. Hence $\|x\| = \|y\|$, and the map $x + M \mapsto \|x\|$ is indeed well-defined.

By the assumption that $\|\cdot\|$ is a seminorm, we have homogeneity and subadditivity for free. Furthermore, the zero element of the quotient vector space X/M is simply the subspace M. Any nonzero element of X/M can be written $x+M$ where $x \notin M$. We then have $x + M \mapsto \|x\| \ne 0$, since $x \notin M = \{x' \in X : \|x'\| = 0\}$ by assumption. It follows that the given map is positive, homogeneous, and subadditive on X/M, and therefore defines a norm.

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