12. $\displaystyle\sum_{n=10}^{\infty} \frac{\sin(n+\frac{1}{2})\pi}{\ln \ln n} = \sum_{n=10}^{\infty} \frac{(-1)^n}{\ln \ln n}$ converges by the alternating series test but only conditionally since $\displaystyle\sum_{n=10}^{\infty} \frac{1}{\ln \ln n}$ diverges to infinity by comparison with $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n}$.
($\ln \ln n < n$ for $n \geq 10$.)

25. Let $u = x - 1$. Then $x = 1 + u$, and

$$\begin{aligned} x \ln x &= (1+u)\ln(1+u) \\ &= (1+u)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} \qquad (-1 < u \le 1) \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^{n+1}}{n}. \end{aligned}$$

Replace n by $n-1$ in the last sum.

$$\begin{aligned} x \ln x &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} + \sum_{n=2}^{\infty}(-1)^{n-2}\frac{u^n}{n-1} \\ &= u + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{n} - \frac{1}{n-1}\right)u^n \\ &= (x-1) + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}(x-1)^n \qquad (0 \le x \le 2). \end{aligned}$$

49. The Fourier sine series of $f(t) = \pi - t$ on $[0, \pi]$ has coefficients

$$b_n = \frac{2}{\pi} \int_0^{\pi} (\pi - t)\sin(nt)\,dt = \frac{2}{n}.$$

The series is $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n}\sin(nt)$.

17. $x = \sin^4 t,\ y = \cos^4 t,\ \left(0 \le t \le \frac{\pi}{2}\right)$.

$$\begin{aligned}
\text{Area } &= \int_0^{\pi/2} (\cos^4 t)(4\sin^3 t \cos t)\, dt \\
&= 4\int_0^{\pi/2} \cos^5 t\,(1-\cos^2 t)\sin t\, dt \qquad \text{Let } u = \cos t \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad du = -\sin t\, dt \\
&= 4\int_0^1 (u^5 - u^7)\, du = 6\left(\frac{1}{6} - \frac{1}{8}\right) = \frac{1}{6} \text{ sq. units.}
\end{aligned}$$

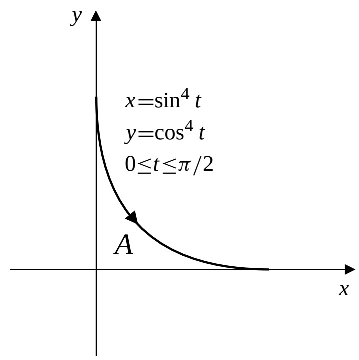

Fig. 4-17

19. a) The density function for the uniform distribution on $[a, b]$ is given by $f(x) = 1/(b-a)$, for $a \le x \le b$. By Example 5, the mean and standard deviation are given by

$$\mu = \frac{b+a}{2}, \qquad \sigma = \frac{b-a}{2\sqrt{3}}.$$

Since $\mu + 2\sigma = \dfrac{b+a}{2} + \dfrac{b-a}{\sqrt{3}} > b$, and similarly, $\mu - 2\sigma < a$, therefore $\Pr(|X-\mu| \ge 2\sigma) = 0$.

b) For $f(x) = ke^{-kx}$ on $[0, \infty)$, we know that $\mu = \sigma = \dfrac{1}{k}$ (Example 6). Thus $\mu - 2\sigma < 0$ and $\mu + 2\sigma = \dfrac{3}{k}$. We have

$$\begin{aligned} \Pr(|X-\mu| \ge 2\sigma) &= \Pr\left(X \ge \frac{3}{k}\right) \\ &= k\int_{3/k}^{\infty} e^{-kx}\,dx \\ &= -e^{-kx}\Big|_{3/k}^{\infty} = e^{-3} \approx 0.050. \end{aligned}$$

c) For $f_{\mu,\sigma}(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{(x-\mu)^2/2\sigma^2}$, which has mean μ and standard deviation σ, we have

$$\begin{aligned} \Pr(|X-\mu| \ge 2\sigma) &= 2\Pr(X \le \mu - 2\sigma) \\ &= 2\int_{-\infty}^{\mu-2\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,dx \\ &\qquad \text{Let } z = \frac{x-\mu}{\sigma} \\ &\qquad dz = \frac{1}{\sigma}\,dx \\ &= \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{-2} e^{-z^2}\,dz \\ &= 2\Pr(Z \le -2) \approx 2 \times 0.023 = 0.046 \end{aligned}$$

from the table in this section.

15.

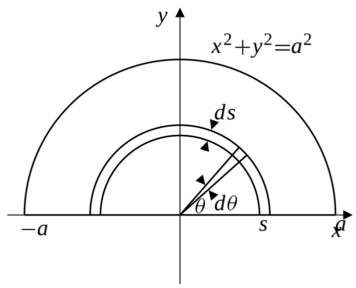

Fig. 4-15

Consider the area element which is the thin half-ring shown in the figure. We have

$$dm = ks\,\pi s\,ds = k\pi\,s^2\,ds.$$

Thus, $m = \dfrac{k\pi}{3}\,a^3$.
Regard this area element as itself composed of smaller elements at positions given by the angle θ as shown. Then

$$\begin{aligned} dM_{y=0} &= \left(\int_0^{\pi} (s\sin\theta)s\,d\theta\right) ks\,ds \\ &= 2ks^3\,ds, \\ M_{y=0} &= 2k\int_0^a s^3\,ds = \frac{ka^4}{2}. \end{aligned}$$

Therefore, $\bar{y} = \dfrac{ka^4}{2} \cdot \dfrac{3}{k\pi a^3} = \dfrac{3a}{2\pi}$. By symmetry, $\bar{x} = 0$. Thus, the centre of mass of the plate is $\left(0, \dfrac{3a}{2\pi}\right)$.

6. $2(x+1)^3 = 3(y-1)^2, \quad y = 1 + \sqrt{\frac{2}{3}}(x+1)^{3/2}$

$$y' = \sqrt{\tfrac{3}{2}}(x+1)^{1/2},$$

$$ds = \sqrt{1 + \frac{3x+3}{2}}\,dx = \sqrt{\frac{3x+5}{2}}\,dx$$

$$L = \frac{1}{\sqrt{2}}\int_{-1}^{0}\sqrt{3x+5}\,dx = \frac{\sqrt{2}}{9}(3x+5)^{3/2}\Big|_{-1}^{0}$$

$$= \frac{\sqrt{2}}{9}\left(5^{3/2} - 2^{3/2}\right) \text{ units.}$$

9. a) About the x-axis:

$$V = \pi \int_0^1 \left(4 - \frac{1}{(1+x^2)^2} \right) dx \qquad \text{Let } x = \tan\theta$$
$$dx = \sec^2\theta \, d\theta$$
$$= 4\pi - \pi \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta} \, d\theta$$
$$= 4\pi - \pi \int_0^{\pi/4} \cos^2\theta \, d\theta$$
$$= 4\pi - \frac{\pi}{2}(\theta + \sin\theta\cos\theta)\Big|_0^{\pi/4}$$
$$= 4\pi - \frac{\pi^2}{8} - \frac{\pi}{4} = \frac{15\pi}{4} - \frac{\pi^2}{8} \text{ cu. units.}$$

b) About the y-axis:

$$V = 2\pi \int_0^1 x \left(2 - \frac{1}{1+x^2} \right) dx$$
$$= 2\pi \left(x^2 - \frac{1}{2}\ln(1+x^2) \right)\Big|_0^1$$
$$= 2\pi \left(1 - \frac{1}{2}\ln 2 \right) = 2\pi - \pi\ln 2 \text{ cu. units.}$$

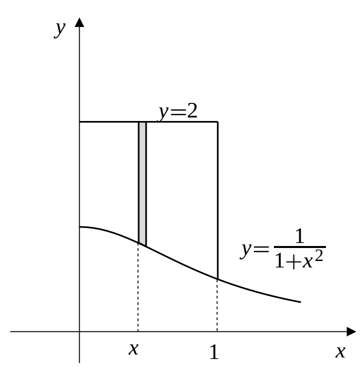

Fig. 1-9

21. $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$ Let $u = \ln x$

$$du = \frac{dx}{x}$$

$$= \int \ln u\,du$$

$$U = \ln u \quad dV = du$$

$$dU = \frac{du}{u} \quad V = u$$

$$= u\ln u - \int du = u\ln u - u + C$$

$$= (\ln x)(\ln(\ln x)) - \ln x + C.$$

27. a) "$\sum a_n$ converges implies $\sum(-1)^n a_n$ converges" is FALSE. $a_n = \dfrac{(-1)^n}{n}$ is a counterexample.

b) "$\sum a_n$ converges and $\sum(-1)^n a_n$ converges implies $\sum a_n$ converges absolutely" is FALSE. The series of Exercise 25 is a counterexample.

c) "$\sum a_n$ converges absolutely implies $\sum(-1)^n a_n$ converges absolutely" is TRUE, because
$|(-1)^n a_n| = |a_n|$.

28. "If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum(a_n + b_n)$" is FALSE. Let $a_n = \dfrac{1}{n}$ and $b_n = -\dfrac{1}{n}$, then $\sum a_n = \infty$ and $\sum b_n = -\infty$ but $\sum(a_n + b_n) = \sum(0) = 0$.

36. a) "If $\lim a_n = \infty$ and $\lim b_n = L > 0$, then $\lim a_n b_n = \infty$" is TRUE. Let R be an arbitrary, large positive number. Since $\lim a_n = \infty$, and $L > 0$, it must be true that $a_n \geq \dfrac{2R}{L}$ for n sufficiently large. Since $\lim b_n = L$, it must also be that $b_n \geq \dfrac{L}{2}$ for n sufficiently large. Therefore $a_n b_n \geq \dfrac{2R}{L}\dfrac{L}{2} = R$ for n sufficiently large. Since R is arbitrary, $\lim a_n b_n = \infty$.

b) "If $\lim a_n = \infty$ and $\lim b_n = -\infty$, then $\lim(a_n + b_n) = 0$" is FALSE. Let $a_n = 1 + n$ and $b_n = -n$; then $\lim a_n = \infty$ and $\lim b_n = -\infty$ but $\lim(a_n + b_n) = 1$.

c) "If $\lim a_n = \infty$ and $\lim b_n = -\infty$, then $\lim a_n b_n = -\infty$" is TRUE. Let R be an arbitrary, large positive number. Since $\lim a_n = \infty$ and $\lim b_n = -\infty$, we must have $a_n \geq \sqrt{R}$ and $b_n \leq -\sqrt{R}$, for all sufficiently large n. Thus $a_n b_n \leq -R$, and $\lim a_n b_n = -\infty$.

d) "If neither $\{a_n\}$ nor $\{b_n\}$ converges, then $\{a_n b_n\}$ does not converge" is FALSE. Let $a_n = b_n = (-1)^n$; then $\lim a_n$ and $\lim b_n$ both diverge. But $a_n b_n = (-1)^{2n} = 1$ and $\{a_n b_n\}$ does converge (to 1).

e) "If $\{|a_n|\}$ converges, then $\{a_n\}$ converges" is FALSE. Let $a_n = (-1)^n$. Then $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} 1 = 1$, but $\lim_{n\to\infty} a_n$ does not exist.

39. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges by the root test of Exercise 31 since

$$\sigma = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$