30. Since $\dfrac{x^2}{x^5+1} \le \dfrac{1}{x^3}$ for all $x \ge 0$, therefore

$$\begin{aligned} I &= \int_0^\infty \frac{x^2}{x^5+1}\,dx \\ &= \int_0^1 \frac{x^2}{x^5+1}\,dx + \int_1^\infty \frac{x^2}{x^5+1}\,dx \\ &\le \int_0^1 \frac{x^2}{x^5+1}\,dx + \int_1^\infty \frac{dx}{x^3} \\ &= I_1 + I_2. \end{aligned}$$

Since I_1 is a proper integral (finite) and I_2 is a convergent improper integral, (see Theorem 2), therefore I converges.

28. $\displaystyle\int \frac{d\theta}{\cos\theta(1+\sin\theta)}$ Let $u=\sin\theta$

$du=\cos\theta\,d\theta$

$$= \int \frac{du}{(1-u^2)(1+u)} = \int \frac{du}{(1-u)(1+u)^2}$$

$$\frac{1}{(1-u)(1+u)^2} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{(1+u)^2}$$

$$= \frac{A(1+2u+u^2)+B(1-u^2)+C(1-u)}{(1-u)(1+u)^2}$$

$$\Rightarrow \begin{cases} A-B=0 \\ 2A-C=0 \\ A+B+C=1 \end{cases} \Rightarrow A=\frac{1}{4},\ B=\frac{1}{4},\ C=\frac{1}{2}.$$

$$\int \frac{du}{(1-u)(1+u)^2}$$

$$= \frac{1}{4}\int \frac{du}{1-u} + \frac{1}{4}\int \frac{du}{1+u} + \frac{1}{2}\int \frac{du}{(1+u)^2}$$

$$= \frac{1}{4}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \frac{1}{2(1+\sin\theta)} + C.$$

20. $I = \int_1^e \sin(\ln x)\,dx$

$$U = \sin(\ln x) \qquad dV = dx$$

$$dU = \frac{\cos(\ln x)}{x}\,dx \qquad V = x$$

$$= x\sin(\ln x)\Big|_1^e - \int_1^e \cos(\ln x)\,dx$$

$$U = \cos(\ln x) \qquad dV = dx$$

$$dU = -\frac{\sin(\ln x)}{x}\,dx \qquad V = x$$

$$= e\sin(1) - \left[x\cos(\ln x)\Big|_1^e + I\right]$$

Thus, $I = \dfrac{1}{2}[e\sin(1) - e\cos(1) + 1]$.

8. A vertical strip has area $dA = 2\left(\frac{a}{\sqrt{2}} - r\right) dr$. Thus, the mass is

$$\begin{aligned} m &= 2\int_0^{a/\sqrt{2}} kr\left[2\left(\frac{a}{\sqrt{2}} - r\right)\right] dr \\ &= 4k\int_0^{a/\sqrt{2}} \left(\frac{a}{\sqrt{2}}r - r^2\right) dr = \frac{k}{3\sqrt{2}}a^3 \text{ g.} \end{aligned}$$

Since the mass is symmetric about the y-axis, and the plate is symmetric about both the x- and y-axis, therefore the centre of mass must be located at the centre of the square.

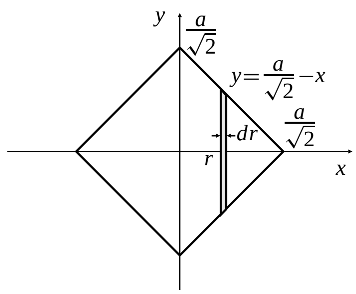

Fig. 4-8