75. $\displaystyle\int \frac{dx}{\tan x + \sin x}$

$$= \int \frac{\cos x\,dx}{\sin x(1+\cos x)} \quad \text{Let } z = \tan(x/2), \quad dx = \frac{2\,dz}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}$$

$$= \int \frac{\dfrac{1-z^2}{1+z^2}\dfrac{2\,dz}{1+z^2}}{\dfrac{2z}{1+z^2}\left(1+\dfrac{1-z^2}{1+z^2}\right)}$$

$$= \int \frac{(1-z^2)\,dz}{z(1+z^2+1-z^2)} = \frac{1}{2}\int \frac{1-z^2}{z}\,dz$$

$$= \frac{1}{2}\ln|z| - \frac{z^2}{4} + C$$

$$= \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\left(\tan\frac{x}{2}\right)^2 + C.$$

Remark: Since

$$\tan^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1-\cos x}{1+\cos x},$$

the answer can also be written as

$$\frac{1}{4}\ln\left|\frac{1-\cos x}{1+\cos x}\right| - \frac{1}{4}\cdot\frac{1-\cos x}{1+\cos x} + C.$$

65. $\displaystyle\int \frac{x^{1/2}}{1+x^{1/3}}\,dx$ Let $x = u^6$

$dx = 6u^5\,du$

$$= 6\int \frac{u^8}{u^2+1}\,du$$

$$= 6\int \frac{u^8+u^6-u^6-u^4+u^4+u^2-u^2-1+1}{u^2+1}\,du$$

$$= 6\int \left(u^6-u^4+u^2-1+\frac{1}{u^2+1}\right)du$$

$$= 6\left(\frac{u^7}{7}-\frac{u^5}{5}+\frac{u^3}{3}-u+\tan^{-1}u\right)+C$$

$$= \frac{6}{7}x^{7/6}-\frac{6}{5}x^{5/6}+2\sqrt{x}-6x^{1/6}+6\tan^{-1}x^{1/6}+C.$$

64. $$\int \frac{x^2}{2x^2-3}\,dx = \frac{1}{2}\int\left(1+\frac{3}{2x^2-3}\right)dx$$
$$= \frac{x}{2}+\frac{\sqrt{3}}{4}\int\left(\frac{1}{\sqrt{2}x-\sqrt{3}}-\frac{1}{\sqrt{2}x+\sqrt{3}}\right)dx$$
$$= \frac{x}{2}+\frac{\sqrt{3}}{4\sqrt{2}}\ln\left|\frac{\sqrt{2}x-\sqrt{3}}{\sqrt{2}x+\sqrt{3}}\right|+C.$$

40. $\displaystyle\int \frac{10^{\sqrt{x+2}}\,dx}{\sqrt{x+2}}$ Let $u = \sqrt{x+2}$

$du = \dfrac{dx}{2\sqrt{x+2}}$

$$= 2\int 10^u\,du = \frac{2}{\ln 10}10^u + C = \frac{2}{\ln 10}10^{\sqrt{x+2}} + C.$$

40. Since $\ln x$ grows more slowly than any positive power of x, therefore we have $\ln x \leq kx^{1/4}$ for some constant k and every $x \geq 2$. Thus, $\dfrac{1}{\sqrt{x}\ln x} \geq \dfrac{1}{kx^{3/4}}$ for $x \geq 2$ and $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}\ln x}$ diverges to infinity by comparison with $\displaystyle\frac{1}{k}\int_2^\infty \frac{dx}{x^{3/4}}$.

34. On [0,1], $\dfrac{1}{\sqrt{x}+x^2} \le \dfrac{1}{\sqrt{x}}$. On $[1,\infty)$, $\dfrac{1}{\sqrt{x}+x^2} \le \dfrac{1}{x^2}$. Thus,

$$\int_0^1 \frac{dx}{\sqrt{x}+x^2} \le \int_0^1 \frac{dx}{\sqrt{x}}$$
$$\int_1^\infty \frac{dx}{\sqrt{x}+x^2} \le \int_1^\infty \frac{dx}{x^2}.$$

Since both of these integrals are convergent, therefore so is their sum $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$.

24. $y(x) = 3 + \int_0^x e^{-y}\,dt \quad \Longrightarrow \quad y(0) = 3$

$\dfrac{dy}{dx} = e^{-y}, \qquad$ i.e. $e^y\,dy = dx$

$e^y = x + C \quad \Longrightarrow \quad y = \ln(x + C)$

$3 = y(0) = \ln C \quad \Longrightarrow \quad C = e^3$

$y = \ln(x + e^3).$

7. The region is a quarter-elliptic disk with semi-axes $a = 2$ and $b = 1$. The area of the region is $A = \pi ab/4 = \pi/2$. The moments about the coordinate axes are

$$M_{x=0} = \int_0^2 x\sqrt{1 - \frac{x^2}{4}}\, dx \quad \text{Let } u = 1 - \frac{x^2}{4}$$

$$du = -\frac{x}{2}\, dx$$

$$= 2\int_0^1 \sqrt{u}\, du = \frac{4}{3}$$

$$M_{y=0} = \frac{1}{2}\int_0^2 \left(1 - \frac{x^2}{4}\right) dx$$

$$= \frac{1}{2}\left(x - \frac{x^3}{12}\right)\bigg|_0^2 = \frac{2}{3}.$$

Thus $\overline{x} = M_{x=0}/A = 8/(3\pi)$ and $\overline{y} = M_{y=0}/A = 4/(3\pi)$. The centroid is $\big(8/(3\pi), 4/(3\pi)\big)$.