# Math 121: Homework 8 solutions

1. (a) For $\sum_{n=0}^{\infty}\frac{1+5^n}{n!}x^n$ we have $R=\lim\frac{1+5^n}{n!}\cdot\frac{(n+1)!}{1+5^{n+1}}=\infty$. The radius of convergence is infinite, the center of convergence is 0. The interval of convergence is the whole real line $(-\infty,\infty)$.

   (b) We have $\sum_{n=1}^{\infty}\frac{(4x-1)^n}{n^n}=\sum_{n=1}^{\infty}(\frac{4}{n})^n(x-1/4)^n$. The center of convergence is $x=1/4$. The radius of convergence is

$$R=\lim\frac{4^n}{n^n}\frac{(n+1)^{n+1}}{4^{n+1}}=\infty.$$

   Hence, the interval of convergence is $(-\infty,\infty)$.

2. (a) Let $x+2=t$, so $x=t-2$. Then

$$\frac{1}{x^2}=\frac{1}{(2-t)^2}=\sum_{n=0}^{\infty}\frac{(n+1)t^n}{2^{n+2}}=\sum_{n=0}^{\infty}\frac{(n+1)(x+2)^n}{2^{n+2}},$$

   $(-4<x<0)$.

   (b) We have

$$\frac{x^3}{1-2x^2}=x^3(\sum_{n=0}^{\infty}(2x^2)^n)=\sum_{n=0}^{\infty}2^nx^{2n+3},$$

   $(-1/\sqrt{2}<x<1/\sqrt{2})$.

   (c) Let $t=x+1$. Then $x=t-1$, and

$$\begin{aligned} e^{2x+3} &= e^{2t+1}=ee^{2t} \\ &= e\sum_{n=0}^{\infty}\frac{2^nt^n}{n!} \quad (forallt) \\ &= \sum_{n=0}^{\infty}\frac{e2^n(x+1)^n}{n!} \quad (forallx) \end{aligned}$$

   (d) Let $t=x-\pi/4$, so $x=t+\pi/4$. Then

$$\begin{aligned} f(x) &= \sin x-\cos x \\ &= \sin(t+\pi/4)-\cos(t+\pi/4) \\ &= \frac{1}{\sqrt{2}}[(\sin t+\cos t)-(\cos t-\sin t)] \\ &= \sqrt{2}\sin t=\sqrt{2}\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!} \\ &= \sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x-/\pi/4)^{2n+1}. \end{aligned}$$

   For all $x$.

(e)

$$\begin{aligned}\ln(e+x^2) &= \ln e + \ln(1+\frac{x^2}{e}) \\ &= \ln e + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{ne^2},\end{aligned}$$

$-\sqrt{e} < x \leq \sqrt{e}$.

(f)

$$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - (x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + ...)$$

$-1 < x < 1$.

3. (a)

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3},$$

for $-1 < x < 1$. Putting $x = 1/\pi$, we get

$$\sum_{n=0}^{\infty}\frac{(n+1)^2}{\pi^n} = \sum_{k=1}^{\infty}\frac{k^2}{\pi^{k-1}} = \frac{1+1/\pi}{(1-1/\pi)^3} = \frac{\pi^2(\pi+1)}{(\pi-1)^3}.$$

(b)

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, \quad -1 < x < 1.$$

Differentiate with respect to $x$ and then replace $n$ by $n+1$:

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}, \quad -1 < x < 1.$$

$$\sum_{n=1}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3}, \quad -1 < x < 1.$$

Now let $x = -1/2$:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n(n+1)}{2^{n-1}} = \frac{16}{27}.$$

Finally, multiply by $-1/2$:

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{n(n+1)}{2^{n}} = -\frac{8}{27}.$$

(c) Since

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \ln(1+x),$$

for $-1 < x \leq 1$, therefore

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n} = \ln(1+1/2) = \ln(3/2).$$

(d)

$$\sum = 2[\frac{x^3}{2} - \frac{1}{3!}(\frac{x^3}{2})^3 + ...] = 2\sin(\frac{x^3}{2})$$

for all $x$.

(e)

$$\sum = \frac{1}{x}\sinh x = \frac{e^x - e^{-x}}{2x},$$

if $x \neq 0$. The sum is 1 if $x = 0$.

(f)

$$\sum = 2[\frac{1}{2} + \frac{1}{2!}(1/2)^2 + \frac{1}{3!}(1/2)^3 + ...] = 2(e^{1/2} - 1).$$

(g)

$$\sum = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}.$$

The series is the Maclaurin series for $\cos x$ with $x^2$ replaced by $x$. For $x > 0$ the series therefore represents $\cos\sqrt{x}$. For $x < 0$, the series is $\sum_{n=0}^{\infty} \frac{|x|^n}{(2n)!}$, which is the Maclaurin series for $\cosh\sqrt{|x|}$.

4. The Fundamental Theorem of Calculus written in the form

$$f(x) = f(c) + \int_c^x f'(t)dt = P_0(x) + E_0(x)$$

is the case $n = 0$ of the above formula. We now apply integration by parts to the integral, setting

$$\begin{aligned} U &= f'(t), \\ dV &= dt \\ dU &= f''(t)dt \\ V &= -(x-t). \end{aligned}$$

We have

$$\begin{aligned} f(x) &= f(c) - f'(t)(x-t)|_{t=c}^{t=x} + \int_c^x (x-t)f''(t)dt \\ &= f(c) + f'(c)(x-c) + \int_c^x (x-t)f''(t)dt \\ &= P_1(x) + E_1(x). \end{aligned}$$

We have now proved the case $n = 1$ of the formula. We complete the proof for general $n$ by mathematical induction. Suppose the formula holds for some $n = k$:

$$f(x) = P_k(x) + E_k(x) = P_k(x) + \frac{1}{k!}\int_c^x (x-t)^k f^{(k+1)}(t)dt.$$

Again we integrate by parts, let

$$\begin{aligned} U &= f^{(k+1)}(t), \\ dV &= (x-t)^k dt, \\ dU &= f^{(k+2)}(t)dt, \\ V &= \frac{-1}{k+1}(x-t)^{k+1}. \end{aligned}$$

We have

$$f(x) = P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!}(x-c)^{k+1} + \frac{1}{(k+1)!}\int_c^x (x-t)^{k+1} f^{(k+2)}(t)dt.$$

If $f(x) = \ln(1+x)$, then

$$\begin{aligned} f'(x) &= \frac{1}{1+x} \\ f''(x) &= \frac{-1}{(1+x)^2}, \\ f'''(x) &= \frac{2}{(1+x)^3}, \\ f^{(4)}(x) &= \frac{-3!}{(1+x)^4}, \ldots.. \\ f^{(n)}(x) &= \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \end{aligned}$$

and

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= -1 \\ f'''(0) &= 2 \\ f^{(4)}(0) &= -3! \\ f^{(n)}(0) &= (-1)^{n-1}(n-1)!. \end{aligned}$$

Therefore, the Taylor Formula is

$$\begin{aligned} f(x) &= x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + .. + \\ &\quad \frac{(-1)^{n-1}(n-1)!}{n!}x^n + E_n(x), \end{aligned}$$

where

$$\begin{aligned} E_n(x) &= \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)dt \\ &= \frac{1}{n!}\int_0^x (x-t)^n \frac{(-1)^n n!}{(1+t)^{n+1}}dt \\ &= (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}dt. \end{aligned}$$

If $0 \le t \le x \le 1$, then $1+t \ge 1$ and

$$|E_n(x)| \le \int_0^x (x-t)^n dt = \frac{x^{n+1}}{n+1} \le \frac{1}{n+1} \to 0$$

as $n \to \infty$.

If $-1 < x \le t \le 0$, then

$$|\frac{x-t}{1+t}| = \frac{t-x}{1+t} \le |x|,$$

because $\frac{t-x}{1+t}$ increases from 0 to $-x = |x|$ as $t$ increases from $x$ to 0. Thus,

$$|E_n(x)| < \frac{1}{1+x}\int_0^{|x|} |x|^n dt = \frac{|x|^{n+1}}{1+x} \to 0$$

as $n \to \infty$ since $|x| < 1$. Therefore,

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n},$$

for $-1 < x \le 1$.

5.

$$\begin{aligned} K(x) &= \int_1^{1+x} \frac{\ln t}{t-1}dt, \quad u = t-1 \\ &= \int_0^x \frac{\ln(1+u)}{u}du \\ &= \int_0^x [1 - \frac{u}{2} + \frac{u^2}{3} - \frac{u^3}{4} + ...]du \\ &= x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + ... \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)^2}, \quad -1 \le x \le 1 \end{aligned}$$

$$
\begin{aligned}
M(x) &= \int_0^x \frac{\tan^{-1}(t^2)}{t^2} dt \\
&= \int_0^x [1 - \frac{t^4}{3} + \frac{t^8}{5} - \frac{t^{12}}{7} + ...] dt \\
&= x - \frac{x^5}{3 \times 5} + \frac{x^9}{5 \times 9} - \frac{x^{13}}{7 \times 13} + ... \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n+1)(4n+1)}, \quad -1 \leq x \leq 1.
\end{aligned}
$$

6. (a)

$$
\begin{aligned}
\lim_{x \to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1 + x^2)} &= \lim_{x \to 0} \frac{(\frac{x^2}{2!} + \frac{x^3}{3!} + ..)^2}{\frac{x^4}{2} - \frac{x^6}{3} + ..} \\
&= \lim_{x \to 0} \frac{\frac{x^4}{4}(1 + x/3 + x^2/12 + ..)^2}{x^4/2 - x^6/3 + x^8/4 - ..} \\
&= \frac{\frac{1}{4}}{\frac{1}{2}} \\
&= \frac{1}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\lim &= \lim_{x \to 0} \frac{\sin x - 1/3! \sin^3 x + 1/5! \sin^5 x - ..}{x[1 - 1/2! \sin^2 x + 1/4! \sin^4 x - .. - 1]} \\
&= \lim_{x \to 0} \frac{-\frac{2}{3!}x^3 + higherdegreeterms}{-\frac{1}{2!}x^3 + higherdegreeterms} \\
&= \frac{2}{3}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
S(x) &= \int_0^x \sin(t^2) dt \\
&= \int_0^x (t^2 - t^6/3! + ...) dt \\
&= \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + ...
\end{aligned}
$$

$$
\lim_{x \to 0} \frac{x^3 - 3S(x)}{x^7} = \frac{x^3 - x^3 + \frac{x^7}{14} - ...}{x^7} = \frac{1}{14}.
$$

(d)

$$\begin{aligned}\lim &= \lim \frac{(x - x + \frac{x^3}{3} - ..)(2x + \frac{4x^2}{2!} + ...)}{2x^2 - 1 + 1 - \frac{4x^2}{2!} + ..} \\ &= \lim \frac{x^4(2/3 + ...)}{x^4(2/3 + ...)} \\ &= 1\end{aligned}$$

7. If $f(x) = \ln(\sin x)$, then calculation of successive derivatives leads to

$$f^{(5)}(x) = 24 \csc^4 x \cot x - 8 \csc^2 x \cot x.$$

Observe that $1.5 < \pi/2 \approx 1.5708$, that $\csc x \geq 1$ and $\cot x \geq 0$, and that both functions are decreasing on that interval. Thus

$$|f^{(5)}(x)| \leq 24 \csc^4(1.5) \cot(1.5) \leq 2$$

for $1.5 \leq x \leq \pi/2$. Therefore, the error in the approximation

$$\ln(\sin 1.5) \approx P_4(x),$$

where $P_4$ is the 4th degree Taylor polynomial for $f(x)$ about $x = \pi/2$, satisfies

$$|error| \leq \frac{2}{5!}|1.5 - \pi/2|^5 \leq 3 \times 10^{-8}.$$

$$\begin{aligned}\int_0^{1/2} e^{-x^4} dx &= \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+1}(4n+1)n!}.\end{aligned}$$

The series satisfies the conditions of the alternating series test, so if we truncate after the term for $n = k - 1$, then the error will satisfy

$$|error| \leq \frac{1}{2^{4k+1}(4k+1)k!}.$$

This is less than 0.000005 if $2^{4k+1}(4k+1)k! > 200000$, which happens if $k \geq 3$. Thus, rounded to five decimal places,

$$\int_0^{1/2} e^{-x^4} dx \approx \frac{1}{2 \cdot 1 \cdot 1} - \frac{1}{32 \cdot 5 \cdot 1} + \frac{1}{512 \cdot 9 \cdot 2} \approx 0.49386.$$

8. $f$ is even, so its Fourier sine coefficients are all zero. Its cosine coefficients are

$$\frac{a_0}{2} = \frac{1}{2} \cdot \frac{2}{3} \int_0^3 f(t)dt = \frac{2}{3}.$$

$$a_n = \frac{2}{3} \int_0^3 f(t) \cos \frac{2n\pi t}{3} dt = \frac{3}{2n^2\pi^2} [\cos(\frac{2n\pi}{3}) - 1 - \cos(2n\pi) + \cos(\frac{4n\pi}{3})].$$

The latter expression was obtained using Maple to evaluate the integral. If $n = 3k$, where $k$ is an integer, then $a_n = 0$. For other integers $n$ we have $a_n = -9/(2\pi^2 n^2)$. Thus the Fourier series of $f$ is

$$\frac{2}{3} - \frac{9}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(\frac{2n\pi t}{3}) + \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(2n\pi t).$$

9. If $f$ is even and has period $T$, then

$$\begin{aligned} b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2n\pi t}{T} dt \\ &= \frac{2}{T} [\int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt + \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt]. \end{aligned}$$

In the first integral in the line above replace $t$ with $-t$. Since $f(-t) = f(t)$ and sine is odd, we get

$$\begin{aligned} b_n &= \frac{2}{T} [\int_{T/2}^{0} f(t)(-\sin \frac{2n\pi t}{T})(-dt) \\ &\quad + \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt] \\ &= \frac{2}{T} [-\int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt + \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt] \\ &= 0. \end{aligned}$$

Similarity, we have

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos \frac{2n\pi t}{T} dt.$$

The corresponding result for an odd function $f$ states that $a_n = 0$ and

$$b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt,$$

and is proved similarly.

10.

$$C_n^0 = \frac{n!}{0!n!} = 1,$$

$$C_n^n = \frac{n!}{n!0!} = 1.$$

If $0 \leq k \leq n$, then

$$\begin{aligned} C_0^{k-1} + C_n^k &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k+1)!}(k + (n-k+1)) \\ &= \frac{(n+1)!}{k!(n+1-k)!} = C_{n+1}^k. \end{aligned}$$