Math 121: Homework 5 solutions

1. Let a circular disk with radius a have center at point $(a, 0)$. Then the disk is rotated about the $y-$axis which is one of its tangent lines. The volume is:

$$V = 2 \times 2\pi \int_0^{2a} x\sqrt{a^2 - (x-a)^2}dx.$$

Let $u = x - a$, $du = dx$. Then we have:

$$\begin{aligned} V &= 4\pi \int_{-a}^{a} (u+a)\sqrt{a^2 - u^2}du \\ &= 4\pi \int_{-a}^{a} u\sqrt{a^2 - u^2}du + 4\pi a \int_{-a}^{a} \sqrt{a^2 - u^2}du \\ &= 0 + 4\pi a(\frac{1}{2}\pi a^2) \\ &= 2\pi^2 a^3. \end{aligned}$$

2. The region is symmetric about $x = y$ so has the same volume of revolution about the two coordinate axes. The volume of revolution about the $y-$axis is

$$V = 2\pi \int_0^8 x(4 - x^{2/3})^{3/2}dx.$$

Let $x = 8\sin^3 u$, $dx = 24\sin^2 u \cos u du$. Thus, we have

$$\begin{aligned} V &= 3072\pi \int_0^{\pi/2} \sin^5 u \cos^4 u du \\ &= 3072\pi \int_0^{\pi/2} (1 - \cos^2 u)^2 \cos^4 u \sin u du. \end{aligned}$$

Let $v = \cos u$, $dv = -\sin u du$. So

$$\begin{aligned} V &= 3072\pi \int_0^1 (1 - v^2)^2 v^4 dv \\ &= 3072\pi \int_0^1 (v^4 - 2v^6 + v^8)dv \\ &= 3072\pi(\frac{1}{5} - \frac{2}{7} + \frac{1}{9}) = \frac{8192\pi}{105}. \end{aligned}$$

3. The volume between height 0 and height z is z^3. Thus,

$$z^3 = \int_0^z A(t)dt,$$

where $A(t)$ is the cross-sectional area at height t. Differentiating the above equation with respect to z, we get $3z^2 = A(z)$. The cross-sectional area at height z is $3z^2$ sq.units.

4. (a)

$$\begin{aligned} V &= 2\int_0^r (2\sqrt{r^2-y^2})^2 dy \\ &= 8\int_0^r (r^2-y^2)dy \\ &= 8(r^2y-\frac{y^3}{3})|_0^r \\ &= \frac{16r^3}{3} cu.units. \end{aligned}$$

(b) The area of an equilateral triangle of base $2y$ is $\frac{1}{2}(2y)(\sqrt{3}y) = \sqrt{3}y^2$. Hence, the solid has volume

$$\begin{aligned} V &= 2\int_0^r \sqrt{3}(r^2-x^2)dx \\ &= 2\sqrt{3}(r^2x-\frac{1}{3}x^3)|_0^r \\ &= \frac{4}{\sqrt{3}}r^3 cu.units. \end{aligned}$$

5. (a) $y = x^2, 0 \le x \le 2, y' = 2x$. So we have

$$L = \int_0^2 \sqrt{1+4x^2}dx.$$

Let $2x = \tan\theta, 2dx = \sec^2\theta d\theta$.

$$\begin{aligned} L &= \frac{1}{2}\frac{x=0}{x=2}\sec^3\theta \\ &= \frac{1}{4}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)|_{x=0}^{x=2} \\ &= \frac{1}{4}(2x\sqrt{1+4x^2}+\ln(2x+\sqrt{1+4x^2}))|_0^2 \\ &= \frac{1}{4}(4\sqrt{17}+\ln(4+\sqrt{17})) \\ &= \sqrt{17}+\frac{1}{4}\ln(4+\sqrt{17}) units. \end{aligned}$$

(b) $y = \ln\frac{e^x-1}{e^x+1}, 2 \le x \le 4,$

$$y' = \frac{2e^x}{e^{2x}-1}.$$

The length of the curve is

$$\begin{aligned} L &= \int_2^4 \sqrt{1+\frac{4e^{2x}}{(e^{2x}-1)^2}}dx \\ &= \int_2^4 \frac{e^{2x}+1}{e^{2x}-1}dx \\ &= \int_2^4 \frac{e^x+e^{-x}}{e^x-e^{-x}}dx = \ln|e^x-e^{-x}||_2^4 \\ &= \ln(e^4-\frac{1}{e^4})-\ln(e^2-\frac{1}{e^2}) \\ &= \ln\frac{e^4+1}{e^2} units. \end{aligned}$$

6.

$$S = 2\pi\int_0^1 |x|\sqrt{1+\frac{1}{x^2}}dx = 2\pi\int_0^1 \sqrt{1+x^2}dx.$$

Let $x=\tan\theta$, $dx=\sec^2\theta d\theta$,

$$\begin{aligned} S &= 2\pi\int_0^{\pi/4}\sec^3\theta d\theta \\ &= \pi(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)|_0^{\pi/4} \\ &= \pi[\sqrt{2}+\ln(\sqrt{2}+1)] sq.units. \end{aligned}$$

7. (a) The mass of the plate is

$$m = 2\int_0^4 ky\sqrt{4-y}dy,$$

let $u=4-y$, $du=-dy$. Then we have

$$\begin{aligned} m &= 2k\int_0^4 (4-u)u^{1/2}du \\ &= 2k(\frac{8}{3}u^{3/2}-\frac{2}{5}u^{5/2})|_0^4 = \frac{256k}{15}. \end{aligned}$$

By symmetry, $M_{x=0}=0$, so $\bar{x}=0$.

$$M_{y=0} = 2\int_0^4 ky^2\sqrt{4-y}dy,$$

let $u=4-y$, $du=-dy$.

$$\begin{aligned} M_{y=0} &= 2k\int_0^4 (16u^{1/2}-8u^{3/2}+u^{5/2})du \\ &= 2k(\frac{32}{3}u^{3/2}-\frac{16}{5}u^{5/2}+\frac{2}{7}u^{7/2})|_0^4 \\ &= \frac{4096k}{105}. \end{aligned}$$

Thus, $\bar{y}=\frac{16}{7}$. The center of mass of the plate is $(0,\frac{16}{7})$.

(b) The mass of the ball is

$$\begin{aligned} m &= \int_{-R}^{R} (y+2R)\pi(R^2-y^2)dy \\ &= 4\pi R(R^2 y - \frac{y^3}{3})|_0^R \\ &= \frac{8}{3}\pi R^4 kg. \end{aligned}$$

By symmetry, the center of mass lies along the y-axis; we need only calculate $\bar{y}$.

$$\begin{aligned} M_{y=0} &= \int_{-R}^{R} y(y+2R)\pi(R^2-y^2)dy \\ &= 2\pi \int_0^R y^2(R^2-y^2)dy \\ &= \frac{4}{15}\pi R^5. \end{aligned}$$

Thus, $\bar{y} = \frac{R}{10}$.

(c) A slice at height z has volume $dV = \pi y^2 dz$ and density kzg/cm^3. Thus, the mass of the cone is

$$\begin{aligned} m &= \int_0^b kz\pi y^2 dz \\ &= \pi k a^2 \int_0^b z(1-z/b)^2 dz \\ &= \pi k a^2 (\frac{z^2}{2} - \frac{2z^3}{3b} + \frac{z^4}{4b^2})|_0^b \\ &= \frac{1}{12}\pi k a^2 b^2 g. \end{aligned}$$

The moment about $z = 0$ is

$$M_{z=0} = \pi k a^2 \int_0^b z^2(1-z/b)^2 dz = \frac{1}{30}\pi k a^2 b^3 g - cm.$$

Thus, $\bar{z} = \frac{2b}{5}$. Hence, the center of mass is on the axis of the cone at height $2b/5$cm above the base.