Math 121: Homework 4 solutions

1. (a)

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{x(1-x)}} &= 2\int_0^{1/2} \frac{dx}{\sqrt{1/4-(x-1/2)^2}} \\ &= 2\lim_{c\to 0_+}\int_c^{1/2} \frac{dx}{\sqrt{1/4-(x-1/2)^2}} \\ &= 2\lim_{c\to 0_+} \sin^{-1}(2x-1)|_c^{1/2} = \pi.\end{aligned}$$

The integral converges.

(b)

$$\begin{aligned}\int_0^{\pi/2} \sec x dx &= \lim_{C\to(\pi/2)_-} \ln|\sec x+\tan x|_0^C \\ &= \lim_{C\to(\pi/2)_-} \ln|\sec C+\tan C| = \infty.\end{aligned}$$

The integral diverges to infinity.

2. (a) Since $0 \le 1-\cos\sqrt{x} = 2\sin^2(\frac{\sqrt{x}}{2}) \le 2(\frac{\sqrt{x}}{2})^2 = \frac{x}{2}$, for $x \ge 0$, therefore

$$\int_0^{\pi^2} \frac{dx}{1-\cos\sqrt{x}} \ge 2\int_0^{\pi^2} \frac{dx}{x},$$

which diverges to infinity.

(b) Since $\sin x \ge \frac{2x}{\pi}$ on $[0,\pi/2]$, we have

$$\begin{aligned}\int_0^\infty \frac{|\sin x|}{x^2}dx &\ge \int_0^{\pi/2} \frac{\sin x}{x^2}dx \\ &\ge \frac{2}{\pi}\int_0^{\pi/2}\frac{dx}{x} = \infty.\end{aligned}$$

The given integral diverges to infinity.

(c) Since $\ln x$ grows more slowly than any positive power of x, therefore we have $\ln x \le kx^{1/4}$ for some constant k and every $x \ge 2$. Thus,

$$\frac{1}{\sqrt{x}\ln x} \ge \frac{1}{kx^{3/4}},$$

for $x \ge 2$ and $\int_2^\infty \frac{dx}{\sqrt{x}\ln x}$ diverges to infinity by comparison with $\frac{1}{k}\int_2^\infty \frac{dx}{x^{3/4}}$.

3. (a) Let $x = \frac{1}{t^2}$, $dx = -\frac{2dt}{t^3}$.

$$\begin{aligned} & \int_1^\infty \frac{dx}{x^2+\sqrt{x}+1} \\ = & \int_1^0 \frac{1}{(\frac{1}{t^2})^2+\sqrt{\frac{1}{t^2}}+1}(-\frac{2dt}{t^3}) \\ = & 2\int_0^1 \frac{tdt}{t^4+t^3+1}. \end{aligned}$$

(b) Let $\sin x = u^2$, $2udu = \cos x dx = \sqrt{1-u^4}dx$.

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} &= 2\int_0^1 \frac{udu}{u\sqrt{1-u^4}} \\ &= 2\int_0^1 \frac{du}{\sqrt{(1-u)(1+u)(1+u^2)}} \\ &= 4\int_0^1 \frac{vdv}{v\sqrt{(2-v^2)(2-2v^2+v^4)}} \\ &= 4\int_0^1 \frac{dv}{\sqrt{(2-v^2)(2-2v^2+v^4)}} \end{aligned}$$

with $1-u = v^2$, $-du = 2vdv$.

(c) One possibility: let $x = \sin\theta$ and get

$$I = \int_{-1}^1 \frac{e^x dx}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} e^{\sin\theta}d\theta.$$

Another possibility:

$$I = \int_{-1}^0 \frac{e^x dx}{\sqrt{1-x^2}} + \int_0^1 \frac{e^x dx}{\sqrt{1-x^2}} = I_1 + I_2.$$

In I_1 put $1+x = u^2$, in I_2 put $1-x = u^2$:

$$\begin{aligned} I_1 &= \int_0^1 \frac{2e^{u^2-1}udu}{u\sqrt{2-u^2}} = 2\int_0^1 \frac{e^{u^2-1}du}{\sqrt{2-u^2}} \\ I_2 &= \int_0^1 \frac{2e^{1-u^2}udu}{u\sqrt{2-u^2}} = 2\int_0^1 \frac{e^{1-u^2}du}{\sqrt{2-u^2}} \end{aligned}$$

so $I = 2\int_0^1 \frac{e^{u^2-1}+e^{1-u^2}}{\sqrt{2-u^2}}du$.

4. Let $x = \frac{1}{t}$, $dx = -\frac{dt}{t^2}$. So we have

$$I = \int_1^\infty e^{-x^2}dx = \int_1^0 e^{-(1/t)^2}(-\frac{1}{t^2})dt = \int_0^1 \frac{e^{-1/t^2}}{t^2}dt.$$

Observe that

$$\begin{aligned}
\lim_{t\to 0_+} \frac{e^{-1/t^2}}{t^2} &= \lim_{t\to 0_+} \frac{t^{-2}}{e^{1/t^2}} \\
&= \lim_{t\to 0_+} \frac{-2t^{-3}}{e^{1/t^2}(-2t^{-3})} \\
&= \lim_{t\to 0_+} \frac{1}{e^{1/t^2}} = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
S_2 &= \frac{1}{3}(\frac{1}{2})[0 + 4(4e^{-4}) + e^{-1}] \\
&\approx 0.1101549
\end{aligned}$$

$$\begin{aligned}
S_4 &= \frac{1}{3}(\frac{1}{4})[0 + 4(16e^{-16}) + 2(4e^{-4}) + 4(\frac{16}{9}e^{-16/9}) + e^{-1}] \\
&\approx 0.1430237
\end{aligned}$$

$$\begin{aligned}
S_8 &= \frac{1}{3}(\frac{1}{8})[0 + 4(64e^{-64} + \frac{64}{9}e^{-64/9} + \frac{64}{25}e^{-64/25} + \\
&\quad \frac{64}{49}e^{-64/49}) + 2(16e^{-16} + 4e^{-4} + \frac{16}{9}e^{-16/9}) + e^{-1}] \\
&\approx 0.1393877.
\end{aligned}$$

Hence, $I \approx 0.14$, accurate to 2 decimal places. These approximations do not converge very quickly, because the fourth derivative of e^{-1/t^2} has very large values for some values of t near 0. In fact, higher and higher derivatives behave more and more badly near 0, so higher order methods cannot be expected to work well either.

5. Let $y = f(x)$. We are given that m_1 is the midpoint of $[x_0, x_1]$ where $x_1 - x_0 = h$. By tangent line approximate in the subinterval $[x_0, x_1]$,

$$f(x) \approx f(m_1) + f'(m_1)(x - m_1).$$

The error in this approximation is

$$E(x) = f(x) - f(m_1) - f'(m_1)(x - m_1).$$

If $f''(t)$ exists for all t in $[x_0, x_1]$ and $|f''(t)| \leq K$ for some constant K, then by Theorem 11 of section 4.9,

$$|E(x)| \leq \frac{K}{2}(x - m_1)^2.$$

Hence,

$$|f(x) - f(m_1) - f'(m_1)(x - m_1)| \leq \frac{K}{2}(x - m_1)^2.$$

We integrate both sides of this inequality. Noting that $x_1 - m_1 = m_1 - x_0 = h/2$, we obtain for the left side

$$\begin{aligned} & |\int_{x_0}^{x_1} f(x)dx - \int_{x_0}^{x_1} f(m_1)dx - \int_{x_0}^{x_1} f'(m_1)(x-m_1)dx| \\ = & |\int_{x_0}^{x_1} f(x)dx - f(m_1)h - f'(m_1)\frac{(x-m_1)^2}{2}|_{x_0}^{x_1}| \\ = & |\int_{x_0}^{x_1} f(x)dx - f(m_1)h|. \end{aligned}$$

Integrating the right side, we get

$$\begin{aligned} \int_{x_0}^{x_1} \frac{K}{2}(x-m_1)^2 dx &= \frac{K}{2}\frac{(x-m_1)^3}{3}|_{x_0}^{x_1} \\ &= \frac{K}{6}(\frac{h^3}{8} + \frac{h^3}{8}) = \frac{K}{24}h^3. \end{aligned}$$

Hence,

$$\begin{aligned} |f_{x_0}^{x_1} f(x)dx - f(m_1)h| &= |\int_{x_0}^{x_1} [f(x) - f(m_1) - f'(m_1)(x-m_1)]dx| \\ &\leq \frac{K}{24}h^3. \end{aligned}$$

$$I = \int_0^1 x^2 dx = \frac{1}{3}.$$

$M_1 = (1/2)^2(1) = 1/4$. The actual error is $I - M_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Since the second derivative of x^2 is 2, the error estimate is

$$|I - M_1| \leq \frac{2}{24}(1-0)^2(1^2) = \frac{1}{12}.$$

Thus the constant in the error estimate for the Midpoint Rule cannot be improved, no smaller constant will work for $f(x) = x^2$.

6. (a) Since $\lim_{t\to\infty} t^{x-1}e^{-t/2} = 0$, there exits $T > 0$ such that $t^{x-1}e^{-t/2} \leq 1$ if $t \geq T$. Thus,

$$0 \leq \int_T^\infty t^{x-1}e^{-t}dt \leq \int_T^\infty e^{-t}dt = 2e^{-T/2}$$

and $\int_T^\infty t^{x-1}e^{-t}dt$ converges by the comparison theorem. If $x > 0$, then

$$0 \leq \int_0^T t^{x-1}e^{-t}dt < \int_0^T t^{x-1}dt$$

converges by Theorem $2(b)$. Thus the integral defining $\Gamma(x)$ converges.

(b)

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \lim_{c\to 0_+, R\to\infty} \int_c^R t^x e^{-t} dt,$$

Let $U = t^x$, $dV = e^{-t}dt$, $dU = xt^{x-1}dx$, and $V = -e^{-t}$,

$$\begin{aligned}\Gamma(x+1) &= \lim_{c\to 0_+, R\to\infty} (-t^x e^{-t}|_c^R + x\int_c^R t^{x-1}e^{-t}dt) \\ &= 0 + x\int_0^\infty t^{x-1}e^{-t}dt = x\Gamma(x).\end{aligned}$$

(c)

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1 = 0!.$$

By (b), $\Gamma(2) = 1\Gamma(1) = 1 \times 1 = 1 = 1!$. In general, if $\Gamma(k+1) = k!$ for some positive integer k, then $\Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1)k! = (k+1)!$. Hence $\Gamma(n+1) = n!$ for all integers $n \geq 0$, by induction.

(d)

$$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2}e^{-t}dt,$$

let $t = x^2$, $dt = 2xdx$, so

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{x}e^{-x^2}2xdx = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$$

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}.$$