Math 121: Homework 3 solutions

1. (a)

$$I = \int \frac{\ln(\ln x)}{x} dx,$$

Let $u = \ln x$, $du = \frac{dx}{x}$. So $I = \int \ln u du$. Let $U = \ln u$, $dV = du$, $dU = \frac{du}{u}$ and $V = u$.

$$\begin{aligned} I &= u \ln u - \int du = u \ln u - u + C \\ &= (\ln x)(\ln(\ln x)) - \ln x + C. \end{aligned}$$

(b)

$$I = \int (\sin^{-1} x)^2 dx.$$

Let $x = \sin\theta$, $dx = \cos\theta d\theta$. So $I = \int \theta^2 \cos\theta d\theta$. Let $U = \theta^2$, $dV = \cos\theta d\theta$, $dU = 2\theta d\theta$ and $V = \sin\theta$.

$$I = \theta^2 \sin\theta - 2\int \theta \sin\theta d\theta,$$

let $U = \theta$, $dV = \sin\theta d\theta$, $dU = d\theta$ and $V = -\cos\theta$.

$$\begin{aligned} I &= \theta^2 \sin\theta - 2(-\theta\cos\theta + \int \cos\theta d\theta) \\ &= \theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta + C \\ &= x(sin^{-1}x)^2 + 2\sqrt{1-x^2}(\sin^{-1} x) - 2x + C. \end{aligned}$$

(c)

$$I = \int x e^x \cos x dx,$$

let $U = x$, $dV = e^x \cos x dx$, $dU = dx$ and $V = \frac{1}{2}e^x(\sin x + \cos x)$.

$$\begin{aligned} I &= \frac{1}{2}xe^x(\sin x + \cos x) - \frac{1}{2}\int e^x(\sin x + \cos x)dx \\ &= \frac{1}{2}xe^x(\sin x + \cos x) - \frac{1}{4}e^x(\sin x - \cos x + \sin x + \cos x) + C \\ &= \frac{1}{2}xe^x(\sin x + \cos x) - \frac{1}{2}e^x \sin x + C. \end{aligned}$$

(d)

$$\begin{aligned} \frac{x^3+1}{x^2+7x+12} &= x - 7 + \frac{37x+85}{(x+4)(x+3)} \\ \frac{37x+85}{(x+4)(x+3)} &= \frac{A}{x+4} + \frac{B}{x+3}, \end{aligned}$$

solving to have $A = 63$ and $B = -26$. Now we have

$$\begin{aligned}\int \frac{x^3+1}{12+7x+x^2}dx &= \int (x-7+\frac{63}{x+4}-\frac{26}{x+3})dx \\ &= \frac{x^2}{2} - 7x + 63\ln|x+4| - 26\ln|x+3| + C.\end{aligned}$$

(e)

$$\begin{aligned}I &= \int \frac{dt}{(t-1)(t^2-1)^2} \\ &= \int \frac{dt}{(t-1)^3(t+1)^2} \\ &= \int \frac{du}{u^3(u+2)^2},\end{aligned}$$

with setting $u = t-1$. Then

$$\frac{1}{u^3(u+2)^2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{u+2} + \frac{E}{(u+2)^2}$$

so we have $A = \frac{3}{16}$, $B = -\frac{1}{4}$, $C = \frac{1}{4}$, $D = -\frac{3}{16}$ and $E = -\frac{1}{8}$.

$$\begin{aligned}I &= \frac{3}{16}\int \frac{du}{u} - \frac{1}{4}\int \frac{du}{u^2} + \frac{1}{4}\int \frac{du}{u^3} - \frac{3}{16}\int \frac{du}{u+2} - \frac{1}{8}\int \frac{du}{(u+2)^2} \\ &= \frac{3}{16}\ln|t-1| + \frac{1}{4(t-1)} - \frac{1}{8(t-1)^2} - \frac{3}{16}\ln|t+1| + \frac{1}{8(t+1)} + C.\end{aligned}$$

(f)

$$\int \frac{dx}{e^{2x}-4e^x+4} = \frac{dx}{(e^x-2)^2} = \int \frac{du}{u(u-2)^2},$$

by setting $u = e^x$.

$$\frac{1}{u(u-2)^2} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{(u-2)^2} = \frac{A(u^2-4u+4)+B(u^2-2u)+Cu}{u(u-2)^2},$$

to have $A = \frac{1}{4}$, $B = -\frac{1}{4}$ and $C = \frac{1}{2}$.

$$\begin{aligned}\int \frac{du}{u(u-2)^2} &= \frac{1}{4}\int \frac{du}{u} - \frac{1}{4}\int \frac{du}{u-2} + \frac{1}{2}\int \frac{du}{(u-2)^2} \\ &= \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u-2| - \frac{1}{2}\frac{1}{(u-2)} + C \\ &= \frac{x}{4} - \frac{1}{4}\ln|e^x-2| - \frac{1}{2(e^x-2)} + C.\end{aligned}$$

(g)

$$I = \int \frac{dx}{x^2(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta d\theta}{\sec^2\theta\tan^3\theta} = \int \frac{\cos^3\theta d\theta}{\sin^2\theta}$$

by setting $x = \sec\theta$, so $dx = \sec\theta\tan\theta d\theta$. Then let $u = \sin\theta$, $du = \cos\theta d\theta$, then

$$\begin{aligned} I &= \int \frac{1-\sin^2\theta}{\sin^2\theta}\cos\theta d\theta \\ &= \int \frac{1-u^2}{u^2}du = -\frac{1}{u} - u + C \\ &= -(\frac{1}{\sin\theta} + \sin\theta) + C \\ &= -(\frac{x}{\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{x}) + C. \end{aligned}$$

(h) Let $x = \tan\theta$, $dx = \sec^2\theta d\theta$.

$$\begin{aligned} I &= \int \frac{dx}{x(1+x^2)^{3/2}} \\ &= \int \frac{\sec^2\theta d\theta}{\tan\theta\sec^3\theta} = \int \frac{\cos^2\theta d\theta}{\sin\theta} \\ &= \int \frac{\cos^2\theta\sin\theta d\theta}{\sin^2\theta} = -\int \frac{u^2du}{1-u^2} = u + \int \frac{du}{u^2-1}, \end{aligned}$$

with $u = \cos\theta$, $du = -\sin\theta d\theta$. We have

$$\frac{1}{u^2-1} = \frac{1}{2}(\frac{1}{u-1} - \frac{1}{u+1}).$$

Thus

$$\begin{aligned} I &= u + \frac{1}{2}\ln|\frac{u-1}{u+1}| + C \\ &= \cos\theta + \frac{1}{2}\ln|\frac{\cos\theta-1}{\cos\theta+1}| + C \\ &= \frac{1}{\sqrt{1+x^2}} + \frac{1}{2}\ln|\frac{\frac{1}{\sqrt{1+x^2}}-1}{\frac{1}{\sqrt{1+x^2}}+1}| + C \\ &= \frac{1}{\sqrt{1+x^2}} + \frac{1}{2}\ln(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}) + C. \end{aligned}$$

(i) Let $x = \tan\frac{\theta}{2}$, $d\theta = \frac{2}{1+x^2}dx$, $\cos\theta = \frac{1-x^2}{1+x^2}$, $\sin\theta = \frac{2x}{1+x^2}$. So

$$\begin{aligned} \int \frac{d\theta}{1+\sin\theta+\cos\theta} &= \int \frac{(\frac{2}{1+x^2})dx}{1+(\frac{1-x^2}{1+x^2})+(\frac{2x}{1+x^2})} \\ &= \int \frac{dx}{1+x} = \ln|1+x| + C = \ln|1+\tan\frac{\theta}{2}| + C. \end{aligned}$$

2. We set $u = \ln x$, $dx = e^u du$.

$$\begin{aligned} I &= \int x^2(\ln x)^4 dx = \int u^4 e^{3u} du \\ &= (a_4u^4 + a_3u^3 + a_2u^2 + a_1u + a_0)e^{3u} + C. \end{aligned}$$

We have

$$\begin{aligned} \frac{dI}{du} &= (4a_4u^3 + 3a_3u^2 + 2a_2u + a_1)e^{3u} + 3(a_4u^4 + a_3u^3 + a_2u^2 + a_1u + a_0)e^{3u} \\ &= u^4e^{3u}. \end{aligned}$$

So $3a_4 = 1$, $3a_3 + 4a_4 = 0$, $3a_2 + 3a_3 = 0$, $3a_1 + 2a_2 = 0$, and $3a_0 + a_1 = 0$. Thus $a_4 = \frac{1}{3}$, $a_3 = -\frac{4}{9}$, $a_2 = \frac{4}{9}$, $a_1 = -\frac{8}{27}$ and $a_0 = \frac{8}{81}$. We now have

$$I = (\frac{1}{3}u^4 - \frac{4}{9}u^3 + \frac{4}{9}u^2 - \frac{8}{27}u + \frac{8}{81})e^{3u} + C$$

$$I = x^3(\frac{(\ln x)^4}{3} - \frac{4(\ln x)^3}{9} + \frac{4(\ln x)^2}{9} - \frac{8\ln x}{27} + \frac{8}{81}) + C.$$

3.

$$\frac{x^5 + x^3 + 1}{(x-1)(x^2-1)(x^3-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1} + \frac{Ex+F}{x^2+x+1}.$$

4. Suppose that $I = \int e^{-x^2}dx = P(x)e^{-x^2} + C$, where P is a polynomial having degree $m \geq 0$. Then we must have

$$\frac{dI}{dx} = (P'(x) - 2xP(x))e^{-x^2} = e^{-x^2}.$$

It follows that $P'(x) - 2xP(x) = 1$. The left side of this equation must be a polynomial of degree $m + 1 \geq 1$ because $2xP(x)$ has degree $m + 1$ and $P'(x)$ only has degree $m - 1$. But the right side of the equation is a polynomial of degree 0. This contradiction shows that no such polynomial $P(x)$ exists.

Since $\frac{d}{dx}Erf(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$ by the Fundamental Theorem of Calculus, we have

$$\int e^{-x^2}dx = \frac{\sqrt{\pi}}{2}Erf(x) + C.$$

Let us try the form

$$J = \int Erf(x)dx = P(x)Erf(x) + Q(x)e^{-x^2} + C,$$

where P and Q are polynomials to be determined. Then

$$\frac{dJ}{dx} = P'(x)Erf(x) + (\frac{2}{\sqrt{\pi}}P(x) + Q'(x) - 2xQ(x))e^{-x^2} = Erf(x).$$

Hence we must have $P'(x) = 1$ and $\frac{2}{\sqrt{\pi}}P(x) + Q'(x) - 2xQ(x) = 0$. The first of these DEs says that $P(x) = x + k$, without loss of generality we can take the constant k to be zero. The second DE says that

$$Q'(x) - 2xQ(x) = -\frac{2x}{\sqrt{\pi}}.$$

The right side has degree 1 and so must the left side. Thus Q must have degree zero. Hence $Q'(x) = 0$ and $Q(x) = \frac{1}{\sqrt{\pi}}$. Therefore

$$J = \int Erf(x)dx = xErf(x) + \frac{1}{\sqrt{\pi}}e^{-x^2} + C.$$

5.

$$I_n = \int_0^{\pi/2} x^n \sin x dx,$$

let $U = x^n$, $dV = \sin x dx$, $dU = nx^{n-1}dx$, $V = -\cos x$.

$$I = -x^n \cos x|_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1} \cos x dx.$$

Let $U = x^{n-1}$, $dV = \cos x dx$, $dU = (n-1)x^{n-2}dx$, $V = \sin x$.

$$\begin{aligned} I &= n[x^{n-1} \sin x|_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2} \sin x dx] \\ &= n(\frac{\pi}{2})^{n-1} - n(n-1)I_{n-2}, \quad (n \geq 2). \end{aligned}$$

$$I_0 = \int_0^{\pi/2} \sin x dx = 1.$$

$$I_6 = 6(\pi/2)^5 - 6(5)\{4(\pi/2)^3 - 4(3)[2(\pi/2) - 2(1)I_0]\} = \frac{3}{16}\pi^5 - 15\pi^3 + 360\pi - 720.$$

Set $U = \sin^{n-1} x$, $dV = \sin dx$, $dU = (n-1)\sin^{n-2} x \cos x dx$, $V = -\cos x$,

$$\begin{aligned} I_n &= \int \sin^n x dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x dx \\ &= -\sin^{n-1} x \cos x + (n-1)(I_{n-2} - I_n) \\ nI_n &= -\sin^{n-1} x \cos x + (n-1)I_{n-2} \\ I_n &= -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}I_{n-2}. \end{aligned}$$

Note that $I_0 = x + C$ and $I_1 = -\cos x + C$, hence

$$\begin{aligned}
I_6 &= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}I_4 \\
&= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}I_2) \\
&= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3\cos x + \frac{5}{8}(-\frac{1}{2}\sin x\cos x + 1/2I_0) \\
&= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + C \\
&= \frac{5x}{16} - \cos x(\sin^5 x/6 + 5\sin^3 x/24 + 5\sin x/16) + C.
\end{aligned}$$

$$\begin{aligned}
I_7 &= -\frac{1}{7}\sin^6 x\cos x + \frac{6}{7}I_5 \\
&= -\frac{1}{7}\sin^6 x\cos x + \frac{6}{7}(-\frac{1}{5}\sin^4 x\cos x + \frac{4}{5}I_3) \\
&= -\frac{1}{7}\sin^6 x\cos x - \frac{6}{35}\sin^4 x\cos x + \frac{24}{35}(-\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}I_1) \\
&= -\frac{1}{7}\sin^6 x\cos x - \frac{6}{35}\sin^4 x\cos x - \frac{8}{35}\sin^2 x\cos x - \frac{16}{35}x + C \\
&= -\cos x(\sin^6 x/7 + 6\sin^4 x/35 + 8\sin^2 x/35 + \frac{16}{35}) + C.
\end{aligned}$$