Homework 6 - Math 321, Spring 2015

Due on Friday February 27

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1. Given a nonconstant non-decreasing function $\alpha : [a, b] \to \mathbb{R}$, let $\mathcal{R}_\alpha[a, b]$ denote the collection of all bounded functions on $[a, b]$ which are Riemann-Stieltjes integrable with respect to α. Is $\mathcal{R}_\alpha[a, b]$ a vector space, a lattice, an algebra?

2. This problem focuses on computing the Riemann-Stieltjes integral for specific choices of integrators.

   (a) Let $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ be a fixed collection of points in $[a, b]$, and let α be an increasing step function on $[a, b]$ that is constant on each of the open intervals (x_{i-1}, x_i) and has jumps of size $\alpha_i = \alpha(x_i+) - \alpha(x_i-)$ at each of the points x_i. For $i = 0$ and n, we make the obvious adjustments

   $$\alpha_0 = \alpha(a+) - \alpha(a), \qquad \alpha_n = \alpha(b) - \alpha(b-).$$

   If $f \in B[a, b]$ is continuous at each of the points x_i, show that $f \in \mathcal{R}_\alpha[a, b]$ and

   $$\int_a^b f \, d\alpha = \sum_{i=0}^{n} f(x_i)\alpha_i.$$

   (b) If f is continuous on $[1, n]$, compute $\int_1^n f(x)d[x]$, where $[x]$ is the greatest integer in x. What is the value of $\int_1^t f(x)d[x]$ if t is not an integer?

3. Determine, with adequate justification, whether each of the following statements is true or false.

   (a) An equicontinuous, pointwise bounded subset of $\mathcal{C}[a, b]$ is compact.

   (b) The function $\chi_{\mathbb{Q}}$ is Riemann integrable on $[0, 1]$.

   (c) The function χ_Δ is Riemann integrable on $[0, 1]$, where Δ denotes the Cantor middle-third set. (We have already run into this set in Homework 2, Problem 5).

   (d) $\bigcap_\alpha \{\mathcal{R}_\alpha[a, b] : \alpha \text{ increasing}\} = \mathcal{C}[a, b]$.

   (e) If f is a monotone function and α is both continuous and non-decreasing, then $f \in \mathcal{R}_\alpha[a, b]$.

   (f) There exists a non-decreasing function $\alpha : [a, b] \to \mathbb{R}$ and a function $f \in \mathcal{R}_\alpha[a, b]$ such that f and α share a common-sided discontinuity.

   (g) If $f \in \mathcal{R}_\alpha[a, b]$ with $m \le f \le M$ and if φ is continuous on $[m, M]$, then $\varphi \circ f \in \mathcal{R}_\alpha[a, b]$.