Homework 4 - Math 321, Spring 2015

Due on Friday February 6

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1. Weierstrass's second theorem states that any continuous 2π-periodic function f on $\mathbb{R}$ is uniformly approximable by trigonometric polynomials. The aim of this exercise is to prove this statement.

   (a) Deduce Weierstrass's second theorem from his first in the special case when f is even. We sketched a proof of this in class, but fill in the details.

   (b) Explain why a verbatim adaptation of the proof does not work if f is odd.

   (c) Given a general function f and a number $\epsilon > 0$, invoke part (a) to find trigonometric polynomials P and Q that approximate the even functions

   $$f(x) + f(-x) \qquad \text{and} \qquad (f(x) - f(-x)) \sin x$$

   to within $\frac{\epsilon}{10}$. Now use P and Q to find a trigonometric polynomial R that uniformly approximates f with error at most ϵ, thereby proving Weierstrass's second theorem.

2. Let A be a normed algebra. If B is a subalgebra of A, conclude that $\overline{B}$ is a subalgebra of A. *Clarification:* Recall that for us, an algebra is a vector space equipped with a vector multiplication that is associative, left and right distributive and compatible with scalars.

3. Given a metric space (X, d), recall that $\mathcal{B}(X)$ is the space of all bounded real-valued functions on X. Let A be a vector subspace of of $\mathcal{B}(X)$. Show that A is a sublattice of $\mathcal{B}(X)$ if and only if $|f| \in A$.

4. For the examples below, explain whether the class of functions $\mathcal{A}$ is dense in $\mathcal{C}(X)$.

   (a) $X = U \times V$, where U and V are compact metric spaces; $\mathcal{A} =$ the class of functions of the form $f(u, v) = g(u)h(v)$ where $g \in \mathcal{C}(U)$, $h \in \mathcal{C}(V)$.

   (b) $X =$ a compact set in $\mathbb{R}^n$; $\mathcal{A} =$ the class of all polynomials in n-variables.

5. Let $X = \{z : |z| = 1\}$ be the unit circle in the complex plane.

   (a) Verify that the space of functions

   $$\mathcal{A} = \left\{ f : f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}, \quad \theta \in [0, 2\pi), \quad c_n \in \mathbb{R} \right\}$$

   is an algebra.

   (b) Show that $\mathcal{A}$ separates points in X and vanishes at no point of X.

   (c) Show that there exist continuous functions on X that cannot be in the uniform closure of $\mathcal{A}$.

   (d) Explain why the statements above do not contradict the Stone-Weierstrass theorem.