Homework 3 - Math 321, Spring 2015

Due on Friday January 30

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1. (a) Show that if f is continuous on $\mathbb{R}$, then there exists a sequence $\{p_n\}$ of polynomials such that $p_n \to f$ uniformly on each bounded subset of $\mathbb{R}$.

   (b) Show that there does not exist a sequence of polynomials converging uniformly on $\mathbb{R}$ to $f(x) = \sin x$.

2. Suppose that f is a continuous function on $[a, b]$ with all vanishing moments, i.e.,

$$\int_a^b x^n f(x)\, dx = 0 \qquad \text{for each } n = 0, 1, 2, \cdots .$$

   Show that $f \equiv 0$.

3. Let $f \in \mathcal{C}[a, b]$ be continuously differentiable, and let $\epsilon > 0$. Show that there is a polynomial p such that

$$||f - p||_\infty < \epsilon \quad \text{ and } \quad ||f' - p'||_\infty < \epsilon.$$

   Use this to conclude that the space $\mathcal{C}^{[1]}[a, b]$ of all functions having a continuous first derivative on $[a, b]$ is separable. The underlying metric on $\mathcal{C}^{[1]}[a, b]$ is generated by the norm $||f||_{\mathcal{C}^{[1]}} = ||f||_\infty + ||f'||_\infty$.

4. Let $\mathcal{P}[a, b]$ denote the space of all polynomials on $[a, b]$. Clearly $\mathcal{P}[a, b] \subseteq \mathcal{C}[a, b]$.

   (a) Show that $\mathcal{P}[a, b]$ is a strict subset of $\mathcal{C}[a, b]$; in other words, there are necessarily nonpolynomial elements in $\mathcal{C}[a, b]$.

   (b) If $f \in \mathcal{C}[a, b]$ is not a polynomial, then show that for any sequence of polynomials $\{p_n\}$ that converges to f uniformly, one must have that $m_n =$ degree of $p_n \to \infty$.

5. Fill in the following steps to arrive at a fact that we used in the proof of the Weierstrass first approximation theorem. Let $B_n(f)$ denote the nth Bernstein polynomial for $f \in \mathcal{C}[0, 1]$, namely

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}.$$

   Set $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

   (a) Show that $B_n(f_0) = f_0$ and $B_n(f_1) = f_1$.

   (b) Show that

$$B_n(f_2) = \left(1 - \frac{1}{n}\right) f_2 + \frac{1}{n} f_1.$$

(c) Use parts (a) and (b) to obtain the relation

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n} \leq \frac{1}{4n}, \quad 0 \leq x \leq 1.$$