Homework 2 - Math 321, Spring 2015

Due on Friday January 23

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1. Let $\mathcal{B}[0,1]$ denote the space of all real-valued bounded functions on $[0,1]$, equipped with the metric topology generated by the sup norm. Show that $\mathcal{B}[0,1]$ is not separable.

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that $\{f_n : n \geq 1\}$ converges uniformly on the set $\mathbb{Q}$ of rationals. Show that $\{f_n : n \geq 1\}$ actually converges uniformly on all of $\mathbb{R}$. (*Hint:* First convince yourself that it suffices to show that $\{f_n\}$ is uniformly Cauchy and then prove it.)

3. Given a set X, let $\mathcal{B}(X)$ denote the vector space of all bounded real-valued functions $f : X \to \mathbb{R}$. Let us endow $\mathcal{B}(X)$ with the sup norm
$$||f||_\infty = \sup_{x \in X} |f(x)|,$$
and its accompanying metric topology.

   (a) Verify that $\mathcal{B}(X)$ is complete. In other words, if $\{f_n\}$ is a Cauchy sequence in $\mathcal{B}(X)$, then show that $\{f_n\}$ converges uniformly to some $f \in \mathcal{B}(X)$. Moreover, show that
   $$\sup_n ||f_n||_\infty < \infty \quad \text{and} \quad ||f_n||_\infty \to ||f||_\infty \text{ as } n \to \infty.$$

   (b) Let $\{g_n\}$ be a sequence in $\mathcal{B}(X)$ satisfying $\sum_{n=1}^\infty ||g_n||_\infty < \infty$. Show that $\sum_{n=1}^\infty g_n$ converges in $\mathcal{B}(X)$ and that
   $$||\sum_{n=1}^\infty g_n||_\infty \leq \sum_{n=1}^\infty ||g_n||_\infty.$$
   This result is often referred to as the Weierstrass M-test.

4. For the series in each of the following examples, determine whether the convergence is uniform, pointwise or neither on the specified interval, with adequate justification.

   (a) the series
   $$\sum_{n=1}^\infty a_n \sin(nx) \quad \text{and} \quad \sum_{n=1}^\infty a_n \cos(nx) \text{ on } \mathbb{R},$$
   where $\sum_{n=1}^\infty |a_n| < \infty$.

   (b) the series
   $$\sum_{n=1}^\infty x^2/(1+x^2)^n \text{ on } |x| \leq 1.$$

5. Recall the space-filling curve we constructed in class, namely

$$x(t) = \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k}t) \quad y(t) = \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k+1}t),$$

where f is the continuous, piecewise linear, even and 2-periodic function that assumes the value 0 on $[0, \frac{1}{3}]$ and 1 on $[\frac{2}{3}, 1]$. Recall also the definition of the standard middle-third Cantor set Δ,

$$\Delta = \bigcap_{n=0}^{\infty} \Delta_n,$$

where $\Delta_0 = [0,1]$, and Δ_n consists of 2^n disjoint closed subintervals obtained by removing the middle-third of the intervals that constitute Δ_{n-1}. Show that $\{(x(t), y(t)) : t \in \Delta\} = [0,1] \times [0,1]$.