Review worksheet - Countability, density, separability

Math 321, Spring 2015

- Two sets A and B are said to be *equivalent* if there exists a bijection $f : A \to B$, i.e., if f is a one-to-one function from A onto B.
- A set A is called *finite* if either $A = \emptyset$ or if A is equivalent to the set $\{1, 2, \cdots n\}$ for some n.
- If A is not finite, it is said to be *infinite*.
- An infinite set is said to be *countably infinite* if it is equivalent to the set $\mathbb{N} = \{1, 2, 3, \cdots\}$ of natural numbers. Thus the elements of a countably infinite set can be *enumerated* or counted according to their correspondence with the natural numbers: $A = \{x_1, x_2, x_3, \cdots\}$ where the x_i are distinct.
- A set is *countable* if it is either finite or countably infinite.

Which of the following statements are true?

1. The set $[0, 1]$ is countable.

2. The countable union of countable sets is countable.

3. The set of rational numbers is countable.

4. The set of real numbers whose decimal expansions contain only 3 and 7 is countable.

- A set A is said to be *dense* in a metric space (M, d) if $\overline{A} =$ closure of $A = M$. More explicitly, A is dense in M if for every $x \in M$ and every $\epsilon > 0$, the following relation holds: $A \cap B(x; \epsilon) \neq \emptyset$. Thus, there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x$.
- A metric space is called *separable* if it contains a countable dense subset.

Which of the following statements are true?

1. Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

2. The space $\ell^2(\mathbb{R})$ of real square-summable sequences, namely

$$\ell^2(\mathbb{R}) = \{x = \{x_n : n \geq 1\} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$$

is separable. Note that $\ell^2(\mathbb{R})$ is a metric space generated by the ℓ^2 norm $||x||_2 = \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$.